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ISAIC 2020 Acceptance and Registration Notification: Paper ID. ISAIC-MS-1086

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Sen, 7 Sep 2020, 18.21

Dear Dr. Gregoria Ariyanti,

Thank you for your support to 2020 International Symposium on Automation, Information and Computing (ISAIC 2020) will be held at Beijing Jiaotong University, Beijing, China, December 2nd-4th, 2020.

This is to inform you that your paper (with ID: **ISAIC-MS-1086, A Note on Eigenvalue of Matrices over The Symmetrized Max-Plus Algebra**) has been accepted for presentation by the conference and will be published in Journal of Physics: Conference Series. Please find attached the formal Invitation Letter, Acceptance Notification, Registration Payment Invoice and Participation Confirmation Form.

Please kindly make payment for the registration and send us the payment voucher together with the filled participation confirmation form by **September 14, 2020**, so we will make arrangements accordingly.

Should you have any questions, please feel free to contact me.

Best regards,

Mandy Chen
Secretary of ISAIC
2020 International Symposium on Automation, Information and Computing (ISAIC 2020)
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Fwd: Proof Confirmation of ISAIC-MS-1086

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Kam, 7 Jan 2021, 11.30

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Date: Sel, 5 Jan 2021 pukul 20.33
Subject: Proof Confirmation of ISAIC-MS-1086
To: ariyantigregoria <ariyantigregoria@gmail.com>

Dear Dr.Gregoria Ariyanti,

Attached is the manuscript formatted based on the IOP Guidelines. Please read it carefully by paying full attention to details, such as proper nouns in italics, symbols, the superscripts and subscripts, etc.

If there are any notes in the proof, please make confirmation and modification accordingly. For any revisions, please mark it in red color and send the updated version back via e-mail no later than January 8(please don't delete any Annotations while revising).

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Gregoria Ariyanti <ariyantigregoria@gmail.com> — Kam, 7 Jan 2021, 20.02
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Dear Dr. Mandy Chen,

Attached is the updated version of the manuscript.
Thank you.

Best regards,
Gregoria Ariyanti

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ISAIC-MS-1086_p...

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Dear Dr. Gregoria Ariyanti,

The revised proof is well received. Thank you for your cooperation. We will let you know once the paper is published, maybe in about 3 months.

Best regards,
Mandy

Mandy Chen
Secretary of ISAIC
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——— Original ———
From: "Gregoria Ariyanti"<ariyantigregoria@gmail.com>;
Date: Thu, Jan 7, 2021 09:02 PM

ACCEPTANCE NOTIFICATION

Thank you for your contribution to 2020 International Symposium on Automation, Information and Computing (ISAIC 2020). This is to inform you that your paper (with ID: **ISAIC-MS-1086**):

Paper Title: *A Note on Eigenvalue of Matrices over The Symmetrized Max-Plus Algebra*
Name of Author(s): *Gregoria Ariyanti**
Affiliation(s): *Department of Mathematics Education, Widya Mandala Surabaya Catholic University, Indonesia*

has been accepted and will be published in *Journal of Physics: Conference Series* which will be indexed by EI Compendex, Scopus, CPCI, etc. We are grateful for your contribution to ISAIC 2020. We also hope that you will continue contributing your excellent work to the future ISAIC conferences.

For any queries, please feel free to contact us.

Best regards,

Mandy Chen
Secretary
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ISAIC 2020 Organizing Committee

A Note on Eigenvalue of Matrices over The Symmetrized Max-Plus Algebra

Gregoria Ariyanti*

Department of Mathematics Education, Widya Mandala Surabaya Catholic University, Indonesia

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Abstract. Max-plus algebra is the structure that doesn't have an inverse of additive. Therefore, there exists an equation that doesn't have a solution. For example, equation $3 \oplus x=2$ has no solution because there is no x such that $max(3,x) = 2$.The max-plus will have an inverse element of addition if that structure is extended to the symmetrized max-plus algebra. The expansion into a larger system is the same as the expansion of the natural number into an integer number.This paper describes the necessary or sufficient condition of the eigenvalue of matrices over the symmetrized max-plus algebra using the linear balance systems $A \otimes x \nabla b$ with ∇ as a balance relation.

1. Introduction

In the max-plus algebra $\mathbb{R}_\varepsilon$, there is a linear equation system one of which form $A \otimes x = b$. Farlow stated that the greatest subsolution of linear system $A \otimes x = b$ is the largest vector x such that $A \otimes x \leq b$ denoted by $x^*(A, b)$[1]. The greatest subsolution is not necessarily a solution of $A \otimes x = b$, so that the linear system does not necessarily have a solution. Therefore, the greatest subsolution is not a sufficient condition for the solution of a linear system over max-plus algebra.

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Each element in $\mathbb{R}_\varepsilon$ does not have an inverse of the $\oplus$, so it can not be defined as a determinant on max-plus algebra. Whereas, every element in the symmetrized max-plus algebra has an inverse to $\oplus$, so it can be defined as a determinant which can then be used in determining the solution of a linear system over the symmetrized max-plus algebra, especially for a square matrix.

With the limitations in $\mathbb{R}_\varepsilon$, which does not have an inverse element in $\oplus$, so $\mathbb{R}_\varepsilon$ extended into the set $\mathbb{S}$ that divided into three parts, they are $\mathbb{S}^\oplus$, $\mathbb{S}^\ominus$,and $\mathbb{S}^\bullet$. Thus, the linear systems over the symmetrized max-plus algebra do not have the equation form but the balanced form. Therefore, the linear systems over $\mathbb{S}$ have the form $A \otimes x \nabla b$ with $A \in M_{m\times n}(\mathbb{S})$, $b \in M_{m\times 1}(\mathbb{S})$, $x \in M_{n\times 1}(\mathbb{S})$, and ∇ as a balance relation. Furthermore, the linear system is called Linear Balance Systems. The purpose of this paper is to determinethe necessary or sufficient condition of the eigenvalue of matrices over the symmetrized max-plus algebra using the linear balance systems $A \otimes x \nabla b$.

In this paper, we will mainly concern linear balance systems over the symmetrized max-plus, especially the homogeneous linear systems. We show that the solution of linear balance systems on $\mathbb{S}^\oplus \cup \mathbb{S}^\ominus \cup \mathbb{S}^\bullet$ is given by the partitioned matrix. Some information about symmetrized max-plus algebra is given in section 2. In Section 3, we discuss the existence of the eigenvalue of matrices over the symmetrized max-plus algebra. The necessary or sufficient conditions of the linear balance systems over $\mathbb{S}$ has a nontrivial solution is given in Section 3.

2. The Symmetrized Max-Plus Algebra

Some basic facts about max-plus algebra and symmetrized max-plus algebra are given in this section based on [2-4], and [5]. Let $\mathbb{R}$ denote the set of all real numbers and $\mathbb{R}_\varepsilon = \mathbb{R} \cup \infty$ with $\varepsilon = -\infty$ as the null element and $e := 0$ as the unit element. For all $a, b \in \mathbb{R}_\varepsilon$, the operations $\oplus$ and $\otimes$ are defined as follows:

$$a \oplus b = max(a, b) \text{ and } a \otimes b = a + b$$

and then, $(\mathbb{R}_\varepsilon, \oplus, \otimes)$ is called the max-plus algebra.

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Definition 2.1. [1,2,6]

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Let $u = (x, y), v = (w, z) \in {\mathbb{R}_\varepsilon}^2$.

1) Two unary operators $\ominus$ and $(.)^\bullet$ are defined as follows: $\ominus u = (y, x)$ and $u^\bullet = u \ominus u$.

2) An element u is called balances with v, denoted by $u \nabla v$, if $x \oplus z = y \oplus w$.

3) A relation $\mathscr{B}$ is defined as follows :

$$(x, y)\mathscr{B}(w, z) \text{ if } \begin{cases} (x, y)\nabla(w, z) if x \neq y and w \neq z \\ (x, y) = (w, z), otherwise \end{cases}.$$

According to De Schutter and De Moor, $\mathscr{B}$ is an equivalence relation based on [1] and [2]. Therefore, we can form a factor set $\mathbb{S} = (\mathbb{R}_\varepsilon^2)/\mathscr{B}$. The structure $(\mathbb{S}, \oplus, \otimes)$ is called the symmetrized max-plus algebra. The addition and multiplication operations on $\mathbb{S}$ are given as follows:

$\overline{(a, b)} \oplus \overline{(c, d)} = \overline{(a \oplus c, b \oplus d)}$ and

$\overline{(a, b)} \otimes \overline{(c, d)} = \overline{(a \otimes c \oplus b \otimes d, a \otimes d \oplus b \otimes c)}$

for $\overline{(a, b)}, \overline{(c, d)} \in \mathbb{S}$. The structure $(\mathbb{S}, \oplus, \otimes)$ is semiring because of $\mathbb{S}$ with $\oplus$ associative, $\mathbb{S}$ with $\otimes$ associative, and $\mathbb{S}$ with $\oplus$ and $\otimes$ satisfies the distributive properties [2].

Lemma 2.2. [1-3]

Given $(\mathbb{S}, \oplus, \otimes)$ be the symmetrized max-plus algebra. The following statements hold.

1) The structure $(\mathbb{S}, \oplus, \otimes)$ satisfies commutative.

2) An element $\overline{(\varepsilon, \varepsilon)}$ is both a null element and an absorbent element.

3) An element $\overline{(e, \varepsilon)}$ is a unit element.

4) The structure $(\mathbb{S}, \oplus, \otimes)$ satisfies idempotent of addition.

The structure $\mathbb{S}$ consists of three classes, that are :

1) $\mathbb{S}^\oplus = \{\overline{(t, \varepsilon)} | t \in \mathbb{R}_\varepsilon\}$ with $\overline{(t, \varepsilon)} = \{(t, x) \in \mathbb{R}_\varepsilon^2 | x < t\}$.

2) $\mathbb{S}^\ominus = \{\overline{(\varepsilon, t)} | t \in \mathbb{R}_\varepsilon\}$ with $\overline{(\varepsilon, t)} = \{(x, t) \in \mathbb{R}_\varepsilon^2 | x < t\}$.

3) $\mathbb{S}^\bullet = \{\overline{(t, t)} | t \in \mathbb{R}_\varepsilon\}$ with $\overline{(t, t)} = \{(t, t) \in \mathbb{R}_\varepsilon^2\}$. The elements of $\mathbb{S}^\bullet$ are called balanced.

The set $\mathbb{S}^\oplus$ is isomorphic with $\mathbb{R}_\varepsilon$. Therefore, it is clear that for $a \in \mathbb{R}_\varepsilon$ can be shown with $\overline{(a, \varepsilon)} \in \mathbb{S}^\oplus$. Furthermore, it is easily shown that for $a \in \mathbb{R}_\varepsilon$ we have :

1) $a = \overline{(a, \varepsilon)}$ where $\overline{(a, \varepsilon)} \in \mathbb{S}^\oplus$.

2) $\ominus a = \ominus \overline{(a, \varepsilon)} = \overline{\ominus (a, \varepsilon)} = \overline{(\varepsilon, a)}$ where $\overline{(\varepsilon, a)} \in \mathbb{S}^\ominus$.

3) $a^\bullet = a \ominus a = \overline{(a, \varepsilon)} \ominus \overline{(a, \varepsilon)} = \overline{(a, a)} \in \mathbb{S}^\bullet$.

Lemma 2.3. [1]

Let $a, b \in \mathbb{R}_\varepsilon$. We have $a \ominus b = \overline{(a, b)}$.

Lemma 2.4. [1]

Let $\overline{(a, b)} \in \mathbb{S}$ with $a, b \in \mathbb{R}_\varepsilon$. The following statements hold :

1) If $a > b$ then $\overline{(a, b)} = \overline{(a, \varepsilon)}$.

2) If $a < b$ then $\overline{(a, b)} = \overline{(\varepsilon, b)}$.

3) If $a = b$ then $\overline{(a, b)} = \overline{(a, a)}$ or $\overline{(a, b)} = \overline{(b, b)}$.

Proof:

1) Let $a > b$. We have that $a \oplus b = a$ or $a \oplus \varepsilon = a \oplus b$. The result that $(a, b)\nabla(a, \varepsilon)$. Its mean that $(a, b)\mathscr{B}(a, \varepsilon)$. Therefore $\overline{(a, b)} = \overline{(a, \varepsilon)}$.

2) Let $a < b$ and we have that $a \oplus b = b$ or $a \oplus b = b \oplus \varepsilon$. The result that $(a, b)\nabla(\varepsilon, b)$. Its mean that $(a, b)\mathscr{B}(\varepsilon, b)$. Therefore $\overline{(a, b)} = \overline{(\varepsilon, b)}$.

3) Let$a = b$and we have that $a \oplus b = b \oplus a$. Its mean that$\overline{(a,b)\nabla(a,a)}$.

So it follows that $(a,b)\mathscr{B}(a,a)$. Thus, $\overline{(a,b)} = \overline{(a,a)}$.

Corollary 2.5. [1]

For $a,b \in \mathbb{R}_\varepsilon$,$a \ominus b = \begin{cases} a\,, if a > b \\ \ominus b, if a < b \\ a^\bullet, if a = b \end{cases}$

Given $\mathbb{S}$ be the symmetrized max-plus algebra, a positive integer n and $M_n(\mathbb{S})$ be the set of all $n \times n$ matrices over $\mathbb{S}$. Operations $\oplus$ and $\otimes$ for matrix over the symmetrized max-plus algebra are given as follows :$C = A \oplus B \Longrightarrow c_{ij} = a_{ij} \oplus b_{ij}$ and $C = A \otimes B \Longrightarrow c_{ij} = \bigoplus_l a_{il} \otimes b_{lj}$.

The nxn zero matrices over $\mathbb{S}$ is ε_n with $(\varepsilon_n)_{ij} = \varepsilon$ and an nxn identity matrix over $\mathbb{S}$ is E_n with

$$(E_n)_{ij} = \begin{cases} e\,, if i = j \\ \varepsilon\,, if i \neq j \end{cases}$$

Definition 2.6.

The matrix $A \in M_n(\mathbb{S})$ is invertible of $\mathbb{S}$ if $A \otimes B \nabla E_n$ and $B \otimes A \nabla E_n$ for any $B \in M_n(\mathbb{S})$.

The properties of balance relation, i.e. the operator ∇, are given in the following lemma.

Lemma 2.7. [2,7]

1. For all $a,b,c \in \mathbb{S}$, $a \ominus c \nabla b$ if and only if $a \nabla b \oplus c$.
2. For all $a,b \in \mathbb{S}^\oplus \cup \mathbb{S}^\ominus$, $a \nabla b \Rightarrow a = b$.

Let $A \in M_n(\mathbb{S})$. The homogeneous linear balance systems $A \otimes x \nabla \varepsilon_{n\times1}$ has a nontrivial solution in $\mathbb{S}^\oplus$ or $\mathbb{S}^\ominus$ if and only if det (A) $\nabla \varepsilon_{n\times1}$.

3. Main Results

Poplin stated that the existence and uniqueness of a solution of the linear balance systems for a square matrix over the symmetrized max-plus algebra $\mathbb{S}^\oplus \cup \mathbb{S}^\ominus$ can be solved by Cramer's rule [6]. Solution with Cramer's rule can be done because every element of the symmetrized max-plus algebra is invertible on $\oplus$ so it can be defined as a determinant of a matrix. The relation between determinant and an adjoint matrix is given in the following lemma.

Lemma 3.1.

Let the symmetrized max-plus algebra$(\mathbb{S},\oplus,\otimes)$ with ε as the null element, e as the unit element, a positive integer n, and $A \in M_n(\mathbb{S})$. Then the following statement holds:

$$\det(A) \otimes E_n \nabla \mathrm{A} \otimes \mathrm{adj(A)} \nabla\ \mathrm{adj(A)} \otimes \mathrm{A}.$$

Poplin stated that if $A \in M_n(\mathbb{S})$and $b \in M_{n\times1}(\mathbb{S})$ then every solution on $\mathbb{S}^\oplus \cup \mathbb{S}^\ominus$ from $A \otimes x \nabla b$ consistent of $\det(A) \otimes x \nabla$ adj(A) $\otimes$ b [6]. Poplin's statement can be explained as follows. According to Lemma 3.1., for $A \in M_n(\mathbb{S})$, $\det(A) \otimes E_n \nabla \mathrm{adj(A)} \otimes \mathrm{A}$, by the linear balance systems $A \otimes x \nabla b$, so $(\det(A) \otimes E_n) \otimes \mathrm{x}\ \nabla(\mathrm{adj(A)} \otimes \mathrm{A}) \otimes \mathrm{x}$.

Furthermore, $\det(A) \otimes (E_n \otimes \mathrm{x})\ \nabla\ \mathrm{adj(A)} \otimes (\mathrm{A} \otimes \mathrm{x})$. Obtainable, $\det(A) \otimes \mathrm{x}\ \nabla\ \mathrm{adj(A)} \otimes \mathrm{b}$. Poplin stated that if it is assumed adj(A) $\otimes$ b has an entry of $\mathbb{S}^\oplus \cup \mathbb{S}^\ominus$ and det A has an inverse, then a solution of Cramer's rule $x^b = (\det A)^{\otimes -1} \otimes adj(A) \otimes b$ is a unique solution with $x \in \mathbb{S}^\oplus \cup \mathbb{S}^\ominus$[6]. While De Schutter and De Moor stated that the homogeneous linear balance systems $A \otimes x \nabla \varepsilon_{n\times1}$ with $A \in M_n(\mathbb{S})$ has a nontrivial solution in $\mathbb{S}^\oplus \cup \mathbb{S}^\ominus$ if and only if $\det A\, \nabla \varepsilon$ [2]. De Schutter, De Moor, and Poplin stated that the given linear balance systems have a solution of $\mathbb{S}^\oplus \cup \mathbb{S}^\ominus$ ([2],[6]). While in this paper, we expand the solution of linear balance systems that on $\mathbb{S}^\oplus \cup \mathbb{S}^\ominus \cup \mathbb{S}^\bullet$. A matrix A can be partitioned by rows and columns, as in the following definition.

Definition 3.2.

Let $A \in M_n(\mathbb{S})$. Partitions of the matrix A are defined as follows:

1) $A_{(n,n)}$ is the $(n-1) \times (n-1)$ matrix obtained by deleting the n-th row and the n-th column of A.

2) $A_{[n,n)}$ is a matrix obtained from the n-th row but is not located on the n-th column of A.

3) $A_{(n,n]}$ is a matrix obtained from the n-th column but is not located on the n-th row of A.
The following example illustrates Definition 3.2.

Example 3.3.

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}$. We have,

$A_{[3,2)} = (a_{31} \quad a_{33})$, $A_{[1,2)} = (a_{11} \quad a_{13})$, $A_{(4,2)} = \begin{pmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix}$, and $A_{(2,1]} = \begin{pmatrix} a_{11} \\ a_{31} \\ a_{41} \end{pmatrix}$.

Lemma 3.4. [4]

For $a \in \mathbb{S}, x, b, d \in M_{n\times 1}(\mathbb{S})$, and $C \in M_{n\times n}(\mathbb{S})$, we have this statement:if $a \otimes x \nabla$ b and $C \otimes x \nabla$ d then $C \otimes b \nabla a \otimes d$.

Each element of the symmetrized max-plus algebra has an inverse to $\otimes$, so it can be defined as the determinant of a matrix over the symmetrized max-plus algebra. The determinant of a matrix over the symmetrized max-plus algebra can be expressed as a determinant of the partition of the matrix, as in the following lemma.

Lemma 3.5. [4]

For a matrix $A \in M_n(\mathbb{S})$

$$\det(A) = det\begin{pmatrix} A_{(n,n)} & A_{(n,n]} \\ A_{[n,n)} & a_{nn} \end{pmatrix} = det(A_{(n,n)}) \otimes a_{nn} \ominus A_{[n,n)} \otimes adj(A_{(n,n)}) \otimes A_{(n,n]}$$

Consequently, the solution of linear balance systems $A \otimes x \nabla$ b can be developed for a square matrix A as in Theorem 3.6.

Theorem 3.6.

Given$A \in M_n(\mathbb{S}), b \in M_{n\times 1}(\mathbb{S})$. A solution$x \in M_{n\times 1}(\mathbb{S})$ of $A \otimes x \nabla$ b satisfies

$$\det(A) \otimes x \nabla \operatorname{adj}(\mathrm{A}) \otimes b$$

Proof:

Suppose $A = \begin{pmatrix} A_{(n,n)} & A_{(n,n]} \\ A_{[n,n)} & a_{nn} \end{pmatrix}, x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, and $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, with $A_{(n,n)}$ is a $(n-1) \times (n-1)$ matrix, $A_{[n,n)}$ is a $1 \times (n-1)$ matrix, and x_1, b_1 is a $(n-1) \times 1$ matrix. Consequently, for the linear balance systems $A \otimes x \nabla$ b we have

$$A_{(n,n)} \otimes x_1 \oplus A_{(n,n]} \otimes x_2 \nabla \mathrm{b}_1 \qquad (1)$$

and

$$A_{[n,n)} \otimes x_1 \oplus a_{nn} \otimes x_2 \nabla \mathrm{b}_2 \qquad (2)$$

From (1), we have $A_{(n,n)} \otimes x_1 \nabla \mathrm{b}_1 \ominus A_{(n,n]} \otimes x_2$. According to Lemma 3.1. we have $\det(A_{(n,n)}) \otimes E_{n-1} \nabla \operatorname{adj}(A_{(n.n)}) \otimes A_{(n,n)}$. Consequently

$$\det(A_{(n,n)}) \otimes x_1 \nabla \operatorname{adj}(A_{(n.n)}) \otimes A_{(n,n)} \otimes x_1.$$

We have

$$\det(A_{(n,n)}) \otimes x_1 \nabla \operatorname{adj}(A_{(n.n)}) \otimes \mathrm{b}_1 \ominus A_{(n,n]} \otimes x_2. \qquad (3)$$

We conclude from (2) that

$$A_{[n,n)} \otimes x_1 \nabla \mathrm{b}_2 \ominus a_{nn} \otimes x_2 \qquad (4)$$

According to Lemma 3.4., the form (3) and the form (4), we have

$$A_{[n,n)} \otimes adj(A_{(n,n)}) \otimes b_1 \ominus A_{(n,n]} \otimes x_2 \nabla \det(A_{(n,n)}) \otimes b_2 \ominus a_{nn} \otimes x_2.$$

Consequently,

$$\det(A_{(n,n)}) \otimes a_{nn} \ominus A_{[n,n)} \otimes adj(A_{(n,n)}) \otimes A_{(n,n]} \otimes x_2 \nabla \det(A_{(n,n)}) \otimes b_2 \ominus A_{[n,n)} \otimes adj(A_{(n,n)}) \otimes b_1 \qquad (5)$$

According to Lemma 3.5. and the form (5), we have

$$det\begin{pmatrix} A_{(n,n)} & A_{(n,n]} \\ A_{[n,n)} & a_{nn} \end{pmatrix} \otimes x_2 \nabla det\begin{pmatrix} A_{(n,n)} & b_1 \\ A_{[n,n)} & b_2 \end{pmatrix} = \left(adj(A_{(n,n)}) \otimes b\right)_2$$

Finally, that det $(A) \otimes x\nabla$ adj(A) $\otimes\ b$. This completes the proof.

The next example shows determining the solution of the linear balance systems.

Example 3.7.

Let A $\otimes\ x\nabla b$ with $A = \begin{pmatrix} 1 & \ominus 3 \\ \ominus 2 & 2^{\bullet} \end{pmatrix}$, and $b = \begin{pmatrix} 2 \\ \ominus 5 \end{pmatrix}$. We have det$(A)$ $= \ominus$ 5, $(adj(A) \otimes b)_1 = det\begin{pmatrix} 2 & \ominus 3 \\ \ominus 5 & 2^{\bullet} \end{pmatrix} = \ominus$ 8, and $(adj(A) \otimes b)_2 = det\begin{pmatrix} 1 & \ominus 2 \\ \ominus 2 & \ominus 5 \end{pmatrix} = \ominus$ 6.

According to Lemma 3.4., we have det $(A) \otimes x\nabla$ adj(A) $\otimes\ b$. In fact, $\ominus$ 5 $\otimes\ x\nabla \begin{pmatrix} 8 \\ 6 \end{pmatrix}$. We have $x\nabla \begin{pmatrix} 3 \\ 1 \end{pmatrix}$. The value x satisfying $A \otimes x\nabla$ b is an element of $\mathbb{S}$. This can be is indicated by taking $x = \begin{pmatrix} 3 \\ 2^{\bullet} \end{pmatrix}$. We have $\begin{pmatrix} 1 & \ominus 3 \\ \ominus 2 & 2^{\bullet} \end{pmatrix} \otimes \begin{pmatrix} 3 \\ 2^{\bullet} \end{pmatrix} = \begin{pmatrix} 5^{\bullet} \\ \ominus 5 \end{pmatrix} \nabla \begin{pmatrix} 2 \\ \ominus 5 \end{pmatrix}$.

Furthermore, we will discuss the existence of eigenvalues of a matrix over the symmetrized max-plus algebra. The necessary and sufficient conditions of the linear balance systems over $\mathbb{S}$ has a nontrivial solution, as stated in the following theorem.

Theorem 3.8.

Given $A \in M_n(\mathbb{S})$. The linear balance systems $A \otimes x\nabla \varepsilon_{n\times 1}$ has a nontrivial solution in $\mathbb{S}$ if and only if det (A) $\nabla\varepsilon$.

Proof:

$(\Longrightarrow)$ Suppose $A = \begin{pmatrix} A_{(n,n)} & A_{(n,n]} \\ A_{[n,n)} & a_{nn} \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, and $\varepsilon = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix}$. From the linear balance systems $A \otimes x\nabla \varepsilon_{n\times 1}$, we have

$$A_{(n,n)} \otimes x_1 \oplus A_{(n,n]} \otimes x_2 \nabla \varepsilon_1 \quad (6)$$

and

$$A_{[n,n)} \otimes x_1 \oplus a_{nn} \otimes x_2 \nabla \varepsilon_2 \quad (7)$$

From (6) we have $A_{(n,n)} \otimes x_1 \nabla \ominus A_{(n,n]} \otimes x_2$.

According to Theorem 3.6., we have

$$\det\left(A_{(n,n)}\right) \otimes x_1 \nabla \ominus \text{adj}\left(A_{(n.n)}\right) \otimes A_{(n,n]} \otimes x_2 \quad (8)$$

From (7) we have

$$A_{[n,n)} \otimes x_1 \nabla \ominus a_{nn} \otimes x_2 \quad (9)$$

According to Lemma 3.4., from (8) and (9) we have

$$A_{[n,n)} \otimes \ominus \text{adj}\left(A_{(n.n)}\right) \otimes A_{(n,n]} \otimes x_2 \nabla \ominus a_{nn} \otimes x_2 \otimes \det\left(A_{(n,n)}\right)$$

The result is

$$\det\left(A_{(n,n)}\right) \otimes a_{nn} \ominus A_{[n,n)} \otimes \text{adj}\left(A_{(n.n)}\right) \otimes A_{(n,n]} \otimes x_2 \nabla \varepsilon \quad (10)$$

Now, according to Lemma 3.5., from (10) we have

$$det\begin{pmatrix} A_{(n,n)} & A_{(n,n]} \\ A_{[n,n)} & a_{nn} \end{pmatrix} \otimes x_2 \nabla \varepsilon \quad (11)$$

Let x is a nontrivial solution, it means that x is not balance with ε, and because of $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, without loss of generality, we can be assume that x_2 is not balance with ε. As a result of (11) we have det $(A)\nabla\varepsilon$.

$(\Longleftarrow)$Suppose $A \otimes x\nabla\varepsilon$ has only the trivial solution, which is x∇ε. As a result, the reduced echelon form of a matrix A does not have a row that balance with ε, so $rank(A) = n$. This means that the matrix A is invertible, so $det(A)$ is not in balance with ε. The result shows that there is a contradiction with the previous result. Thus, the linear balance systems $A \otimes x\nabla\varepsilon$ has a nontrivial solution.

The eigenvalues of a matrix in the symmetrized max-plus algebra are defined as follows.

Definition 3.9.

Let $A \in M_n(\mathbb{S})$. $\lambda \in \mathbb{S}$ is called eigenvalues of A if there is $v \in M_{n\times 1}(\mathbb{S})$, v is not in balance with $\varepsilon_{n\times 1}$ such that $A \otimes v \nabla \lambda \otimes v$. The vector v is called the eigenvectors of A corresponding to λ.

Furthermore, the characteristics form of a matrix in the symmetrized max-plus algebra is given in Definition3.10.

Definition 3.10. [2]

Let $A \in M_n(\mathbb{S})$. The characteristic form of A is defined as $\det(A \ominus \lambda \otimes E_n) \nabla \varepsilon$.

According to Theorem 3.8. and Definition 3.10, the following necessary and sufficient conditions developed eigenvalues of a matrix in the symmetrized max-plus algebra.

Theorem 3.11.

Let$A \in M_n(\mathbb{S})$. Scalar $\lambda \in \mathbb{S}$ is an eigenvalue of A if and only if λsatisfies the characteristic form $\det(A \ominus \lambda \otimes E_n)\nabla\varepsilon$.

Proof:

($\Longrightarrow$)Since$\lambda \in \mathbb{S}$ is an eigenvalue of A, from Definition 3.9., for $A \otimes v \nabla \lambda \otimes v$, we have that $A \otimes v \nabla \lambda \otimes E_n \otimes v$, or $\lambda \otimes E_n \otimes v \nabla A \otimes v$. Consequently, $(\lambda \otimes E_n \otimes v \ominus A \otimes v)\nabla\varepsilon$, so we have $(\lambda \otimes E_n \ominus A) \otimes v \nabla \varepsilon$. Consequently, according to Theorem 3.8., we have $\det(A \ominus \lambda \otimes E_n)\nabla\varepsilon$.

($\Longleftarrow$)Sinceλ satisfies the characteristic form $\det(A \ominus \lambda \otimes E_n)\nabla\varepsilon$, so according to Theorem 3.8., there exists the linear balance systems$(A \ominus \lambda \otimes E_n) \otimes v \nabla \varepsilon_{n\times 1}$ that have a nontrivial solution in $\mathbb{S}$. Consequently $A \otimes v \nabla \lambda \otimes v$, so according to Definition 3.9., $\lambda \in \mathbb{S}$ is an eigenvalue for A.

For an invertible matrix, we can show that ε is not an eigenvalue, and conversely, as discussed on Lemma 3.12.

Lemma 3.12.

A matrix $A \in M_n(\mathbb{S})$ is invertible if and only if ε is not an eigenvalue for A.

Proof:

($\Longrightarrow$) Consider A is an invertible matrix. Assume that ε is an eigenvalue for A, then $A \otimes v \nabla \varepsilon \otimes v$. Consequently, $A \otimes v \nabla \varepsilon$. According to Theorem 3.8., $\det(A)\nabla\varepsilon$, so, we have A is not an invertible matrix, and show that this leads to a contradiction. Thus ε is not an eigenvalue for A.

($\Longleftarrow$) Proof by contrapositive. Since A is not an invertible matrix, consequently $\det(A)\nabla\varepsilon$. Furthermore, according to Theorem 3.8., because $\det(A)\nabla\varepsilon$ so the linear balance systems $A \otimes v \nabla \varepsilon_{n\times 1}$ has a nontrivial solution, or, equivalently $A \otimes v \nabla \varepsilon \otimes v$. Thus ε is an eigenvalue for A. This completes the proof.

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