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The Discrete Lyapunov Equation of The Orthogonal Matrix in Semiring

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Abstract. Semiring is an algebraic structure of $(S, +, \times)$. Similar to a ring, but without the condition that each element must have an inverse to the adding operation. The forms $(S, +)$ and $(S, \times)$ are semigroups that satisfy the distributive law of multiplication and addition. In matrix theory, there is a term known as the Kronecker product. This operation transforms two matrices into a larger matrix containing all possible products of the entries in the two matrices. This Kronecker product has several properties often used to solve the complex problems of linear algebra and its applications. The Kronecker product is related to the Lyapunov equation of a linear system. Based on previous research in the Lyapunov equation in conventional linear algebra, this paper will describe the characteristics of the Lyapunov equation in a semiring linear system in terms of the Kronecker product.

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Key Words and Phrases: Linear system, the Lyapunov equation, semiring, the Kronecker product

1. Introduction

A non-empty set G with a binary operation $*$ is known as Group if it has the following properties: associative, a zero element of binary operations $*$, and every element, not a zero element, has an inverse. Meanwhile, a non-empty set R with two binary operations, particularly $*$ and $\circ$ is called Ring. If it has the following properties: $(R, *)$ is a commutative group, $(R, \circ)$ is closed, $(R, \circ)$ is associative, and distributive. If the Ring has the following properties: commutative to binary operation $\circ$, it has a unit element of binary operation $\circ$, and every element, not a zero element, has an inverse to binary operation $\circ$, it is called Field. Different algebraic systems will appear if Group and Ring conditions are weakened, particularly Semigrup and Semiring. If a few Group or Ring characteristics are removed, the algebraic structure formed is Semigrup, after which Semiring. One of

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the problems and applications often encountered in mathematics is completing the Linear Equations System.

The Kronecker product is a binary matrix operator that maps two arbitrarily dimensioned matrices into a larger matrix with a particular block structure ([14]). The Kronecker product discussed by Zhou et al. (1996) and Whitcomb (2020) applies to matrices whose elements are real numbers ([4],[11]). The set of real numbers is a field.

The linear equations system that researchers have developed is a system of linear equations over Field, which include real numbers R or complex numbers C ([12],[4]). In other studies, the research object is extended not to Field anymore but to commutative Ring and linear equations system Commutative over ring have been discussed by Brown ([4]). Likewise, assuming an extension from the Ring to Ring commutative does not generally change the definition.

This paper presents the characteristics of the discrete Lyapunov equation of a matrix. The scope of the topic is a system of linear equations in semiring in terms of the Kronecker product. First, section 2 will review some basic facts about the semiring, the Kronecker product of the matrix in semiring, and the Lyapunov equation of the matrix in semiring. Then, in Section 3, we show a necessary or sufficient condition of the Lyapunov equation in a linear system over semiring in terms of the Kronecker product.

2. Materials and literature review

2.1. Semiring and Matrices in Semiring

Definition 1. *Semigroup S is an empty set equipped with an associative binary $*$ operation, $x * (y * z) = (x * y) * z$ for every $x, y, z \in S$.*

Poplin defines a semiring and its properties as follows([2],[13],[12],[11]).

Definition 2. *Semiring is a non-empty set S with two binary operations, addition $(+)$ and multiplication $(\times)$, which have the following properties: commutative and associative properties of $+$, associative property of $\times$, distributive property of $\times$ to $+$, the set S has a zero element $0 \in S$ so that $0 + a = a + 0 = a$ and $0 \times a = a \times 0 = 0$ for every $a \in S$. This zero element is called the absorbent element (absorption), and the set S has a unit element e, $e \times a = a \times e = a$ for every $a \in S$.*

The commutative and idempotent characteristics in Group and Ring also apply in Semiring ([13]). Let $M_{n\times 1}(S)$ be the set of all vectors $n \times 1$ with the elements of Semiring S. And, let $M_{n\times n}(S)$ be the set of all $n \times n$ matrices with the elements of semiring S ([4],[1]). The $+$ and $\times$ operation for matrices over Semiring is defined as in Definition 3.

Definition 3. *Let S Semiring, a positive integer n, and $M_n(S)$ is the set of all $n \times n$ matrices over S. For every $A, B \in M_n(S)$, $+$ and $\times$ operations over Semiring S are defined $C = A + B$ as $c_{ij} = a_{ij} + b_{ij}$ and $C = A \times B$ as $c_{ij} = \sum_l a_{il} \times b_{lj}$.*

The Semiring S has 0 as a zero element and 1 as an identity element, as in the matrix of conventional algebra. We can form a zero matrix and an identity matrix based on the zero element and the identity element ([2]). The zero matrix $n \times n$ over Semiring S is 0_n and is defined as a matrix with all elements equal to the $0-$element, that is $(0_n)_{ij} = 0$. The identity matrix $n \times n$ over S is defined as the matrix with all elements equal to the $e-$element, that is, $[I_n]_{ij} = \begin{cases} e, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$.
In Semiring, the element of Semiring has an inverse operation on + to determine a matrix determinant in Semiring S. A permutation characterizes a determinant of a matrix over a Semiring S.

2.2. Kronecker Product

Kronecker's product is related to the stack operator. The stack operator maps an $n \times m$ matrix to an $nm \times 1$ vector ([14]). The stack of the $n \times m$ matrix A is represented by $vec(A)$, a vector formed by stacking the columns of A on the vector $nm \times 1$.

Example 1. *Let A is a matrix with* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, *then its stack form is*

$$vec(A) = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}.$$

If C is an $n \times m$ matrix comprising m column vectors $c_1, c_2, c_3, ..., c_m$, where each c_i is an $n \times 1$ vector $C = [c_1, c_2, c_3, ..., c_m]_{n \times m}$, then

$$vec(C) = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix}_{nm \times 1}.$$

Let $X \in M_{m \times n}(S)$, the form $vec(X)$ denote the vector formed by stacking the columns of X into one long vector:

$$vec(X) = \begin{bmatrix} x_{11} & x_{21} & \dots & x_{m1} & x_{12} & x_{22} & \dots & x_{1n} & x_{2n} & \dots & x_{mn} \end{bmatrix}^T.$$

Kronecker product is an operation on two matrices that do not require size ([8], [9]). The notation $\otimes$ denotes Kronecker products. With S semiring, let $A \in M_{m \times n}(S)$ and $B \in M_{p \times q}(S)$, then the Kronecker product of A and B is defined as

$$A \otimes B := \begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{bmatrix} \in M_{mp \times nq}(S).$$

Furthermore, if A and B are square matrices with $A \in M_{n\times n}(S)$ and $B \in M_{m\times m}(S)$ then the Kronecker sum of A and B is defined as

$$A \oplus B := (A \otimes I_m + I_n \otimes B) \in M_{nm\times nm}(S).$$

Properties of Kronecker product is given the following theorem ([8], [14]).

Theorem 1. *For $A \in M_{m\times n}(S)$ and $B \in M_{p\times q}(S)$ with S semiring, we have the following properties: the Kronecker product is associative, not in general commutative, that is, $(A\otimes B) \neq (B\otimes A)$, and transpose distributes over the Kronecker product (do not reverse order) $(A \otimes B)^T = A^T \otimes B^T$.*

Let $A \in M_{n\times n}(S)$ and $B \in M_{m\times m}(S)$, and λ_i with $i = 1, 2, ..., n$ be the eigenvalues of A and μ_i with $i = 1, 2, ..., m$ be the eigenvalues of B. Then we have the following properties: the eigenvalues of $A \otimes B$ are the mn numbers $\lambda_i\mu_j$, and the eigenvalues of $A\oplus B = (A\otimes I_m)+(I_n\otimes B)$ are the mn numbers $\lambda_i+\mu_j$, with $i = 1, 2, ..., n, j = 1, 2, ..., m$.

2.3. Discrete Lyapunov equation

The linear system is closely related to stability, which can be observed using the eigenvalue criterion of matrix A. Furthermore, the stability of the linear system is closely related to the existence of a solution to the Lyapunov equation ([6]). Therefore, this method can determine the system's stability without knowing the system's solution. Lyapunov's equation for a linear system over a field was given by Zhou ([7]). In this study, the Lyapunov equation is defined for a discrete linear system on a semiring, adopting the meaning of the Lyapunov equation for a linear system on a plane. The discrete Lyapunov equation for the linear system over semiring is defined as follows ([10], [5]).

Definition 4. *Given a matrix $A, X, Q \in M_n(S)$. The Lyapunov equation for a linear system over a semiring is defined as $AXA^T - X + Q = 0$.*

For linear systems over the field, the existence of solutions to the Lyapunov equations is associated with asymptotic stability. The system is asymptotically stable if a solution to the Lyapunov equation exists. On the other hand, if the system is asymptotically stable, a solution to the Lyapunov equation exists. ([6]).

3. Results and discussion

The problem of discrete Lyapunov equations over semirings is limited to orthogonal matrices. These are because of the limited nature of the semiring.

Theorem 2. *Let S is semiring. Then for any matrices $A \in M_{k\times m}(S), B \in M_{n\times l}(S)$, and $X \in M_{m\times n}(S)$, we have $vec(AXB) = (B^T \otimes A)vec(X)$.*

Proof. We have $(AXB)_{.k} = \sum_j b_{jk} AX_{.j} = \begin{pmatrix} b_{1k}A & b_{2k}A & \dots & b_{nk}A \end{pmatrix}$ and

$$\begin{bmatrix} X_{.1} \\ X_{.2} \\ \vdots \\ X_{.n} \end{bmatrix} = \left[B_{.k}^T \otimes A\right] vec(X) = \left[(B^T)_{k.}^T \otimes A\right] vec(X).$$

Furthermore, we conclude $vec(AXB) = [B^T \otimes A]vec(X)$ since the transpose of the kth column of B is the kth row of B^T.

In the following, another property of the Kronecker product for matrix over a semiring is given.

Theorem 3. *Let $A \in M_{m\times m}(S), B \in M_{n\times n}(S)$, and $X \in M_{m\times n}(S)$, we have $vec(AX + XB) = (B^T \oplus A)vec(X)$.*

Proof. From the definition of Kronecker sum, we have

$$(B^T \oplus A)vec(X) = (B^T \otimes I_m + I_n \otimes A)vec(X) = (B^T \otimes I_m)vec(X) + (I_n \otimes A)vec(X).$$

According to Theorem 2, we have

$$(B^T \oplus A)vec(X) = vec(I_m XB) + vec(AXI_n^T) = vec(XB) + vec(AX).$$

Example 2. *As is well known, the max-plus algebra $\mathbb{R}_\epsilon$ is semiring. Let A, B, and X matrices over max-plus algebra, with $A = \begin{bmatrix} 3 & \epsilon & 1 \end{bmatrix}$, $B = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$, and $X = \begin{bmatrix} 2 & 5 \\ 1 & 2 \\ \epsilon & -3 \end{bmatrix}$. Based on binary operations on max-plus algebra, as given in Ariyanti ([3]), we have*

$$\begin{aligned} AXB &= \begin{bmatrix} 3 & \epsilon & 1 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 2 \\ \epsilon & -3 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 3 & \epsilon & 1 \end{bmatrix} \begin{bmatrix} 7 \\ 6 \\ -3 \end{bmatrix} \\ &= \begin{bmatrix} 10 \end{bmatrix} \end{aligned}$$

Because $AXB = \begin{bmatrix} 10 \end{bmatrix}$, we get $vec(AXB) = \begin{bmatrix} 10 \end{bmatrix}$. Next, we have

$$\begin{aligned} (B^T \otimes A)vec(X) &= \left(\begin{bmatrix} 5 & 0 \end{bmatrix} \otimes \begin{bmatrix} 3 & \epsilon & 1 \end{bmatrix}\right) \begin{bmatrix} 2 & 5 & 1 & 2 & \epsilon & -3 \end{bmatrix}^T \\ &= \begin{bmatrix} 8 & \epsilon & 6 & 3 & \epsilon & 1 \end{bmatrix} \begin{bmatrix} 2 & 5 & 1 & 2 & \epsilon & -3 \end{bmatrix}^T = \begin{bmatrix} 10 \end{bmatrix} \end{aligned}$$

Due to the limitations of semirings, the existence of discrete solutions to the Lyapunov equations on semirings depends on the orthogonal matrix. This statement is given in the following theorem.

Theorem 4. *Let discrete Lyapunov equation over semiring* $AXA^T - X + Q = 0$ *with* $A \in M_{n\times n}(S), Q \in M_{n\times n}(S)$. *For* A *is orthogonal matrice, there is a unique solution* $X \in M_{n\times n}(S)$ *if and only if* $\lambda_i(A) + \lambda_j(-A) \neq 0$.

Proof. A matrice A is orthogonal if $A^T = A^{-1}$; hence we have $AXA^T - X + Q = 0$ and then $(AXA^T)A - XA = -QA$. Because A is orthogonal, we have $AX - XA = -QA$. Therefore, $AX + X(-A) = Q(-A)$ and $vec(AX + X(-A)) = ((-A)^T \oplus A)vec(X)$. Finally, we have $((-A)^T \oplus A)vec(X) = vec(-QA)$. Based on conditions, we have a unique solution if and only if $(-A)^T \oplus A$ non-singular.

4. Conclusion

We show that the solutions of the discrete Lyapunov equations for matrices over semirings are also valid. The linear system that has been developed is the semiring linear system. Due to its semiring nature, not all elements have inverses. It is necessary to have a particular case, namely by reviewing the Lyapunov equation of the system. In addition, it is required to relate it to the kroner product as given in Theorem 3 and Theorem 4. Furthermore, this condition can be used to find other characteristics of a linear system over a semiring.

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