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A NOTE OF THE LINEAR BALANCE SYSTEMS FOR MATRIX X THAT SATISFIES $A \otimes X \otimes A \nabla A$

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Abstract

A linear system over the symmetrized max plus algebra has form $A \otimes x \nabla b$ with ∇ as a balance relation. The linear system is called the linear balance systems. This paper describes the necessary and sufficient condition of a solution of the linear balance systems with a matrix X that satisfies $A \otimes X \otimes A \nabla A$. We obtain that if X is any matrix satisfying $A \otimes X \otimes A \nabla A$, then $A \otimes x \nabla b$ has a solution if and only if $A \otimes X \otimes b \nabla b$, in which case the most general solution is $x = X \otimes b \oplus (E \ominus X \otimes A) \otimes h$, where h is arbitrary and $A \in M_{m \times n}(\mathbb{S})$.

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1. Introduction

Each element in $\mathbb{R}_\varepsilon$ does not have an inverse of the $\oplus$, so it cannot be defined as a determinant on max plus algebra. Whereas, every element in the symmetrized max plus algebra has an inverse to $\oplus$, so it can be defined as a determinant which can then be used in determining the solution of a linear system over the symmetrized max plus algebra, especially for a square matrix.

In the max plus algebra $\mathbb{R}_\varepsilon$, there is a linear equation system one of which is in the form $A \otimes x = b$. Farlow [3] stated that the greatest subsolution of linear system $A \otimes x = b$ is the largest vector x such that $A \otimes x \leq b$ denoted by $x^*(A, b)$. The greatest subsolution is not necessarily a solution of $A \otimes x = b$, so that the linear system does not necessarily have solution. Therefore, the greatest sub solution is not a sufficient condition for the solution of linear system over the max plus algebra.

With the limitations in $\mathbb{R}_\varepsilon$, which does not have an inverse element in $\oplus$, so $\mathbb{R}_\varepsilon$ extended into the set $\mathbb{S}$ that divided into three parts, they are $\mathbb{S}^{\oplus}$, $\mathbb{S}^{\ominus}$, and $\mathbb{S}^{\bullet}$. Thus, the linear system over the symmetrized max plus algebra does not have the equation form but the balance form. Therefore, the linear systems over $\mathbb{S}$ has the form $A \otimes x \nabla b$ with $A \in M_{m\times n}(\mathbb{S})$, $b \in M_{m\times 1}(\mathbb{S})$, $x \in M_{n\times 1}$ and ∇ as a balance relation. Furthermore, the linear system is called the *linear balance systems*. The purpose of this paper is to determine the condition of a solution of the linear balance systems with a matrix X that satisfies $A \otimes X \otimes A \nabla A$.

2. The Symmetrized Max Plus Algebra

Let $\mathbb{R}$ denote the set of all real numbers and $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{\varepsilon\}$ with $\varepsilon := -\infty$ as the null element and $e := 0$ as the unit element. For all $a, b \in \mathbb{R}_\varepsilon$, the operations $\oplus$ and $\otimes$ are defined as follows:

$$a \oplus b = \max(a, b) \text{ and } a \otimes b = a + b$$

and then, $(\mathbb{R}_\varepsilon, \oplus, \otimes)$ is called the *max plus algebra*.

Definition 2.1 [2, 4]**.** Let $u = (x, y), v = (w, z) \in R_\varepsilon^2$.

(1) Two unary operators $\ominus$ and $(.)^\bullet$ are defined as follows:

$$\ominus u = (y, x) \text{ and } u^\bullet = u \oplus (\ominus u).$$

(2) An element *u* is called *balances* with *v*, denoted by $u \nabla v$, if

$$x \oplus z = y \oplus w.$$

(3) A relation $\mathcal{B}$ is defined as follows:

$$(x, y)\mathcal{B}(w, z) \text{ if } \begin{cases} (x, y)\nabla(w, z), \text{ if } x \neq y \text{ and } w \neq z, \\ (x, y) = (w, z), \text{ otherwise.} \end{cases}$$

Because $\mathcal{B}$ is an equivalence relation, we have the set of factor $\mathbb{S} = \mathbb{R}_\varepsilon^2/\mathcal{B}$ and the system $(\mathbb{S}, \oplus, \otimes)$ is called the *symmetrized max plus algebra*, with the operations of addition and multiplication on $\mathbb{S}$ are defined as follows:

$$\overline{(a, b)} \oplus \overline{(c, d)} = \overline{(a \oplus c, b \oplus d)},$$

$$\overline{(a, b)} \oplus \overline{(c, d)} = \overline{(a \otimes c \oplus b \otimes d, a \otimes d \oplus b \otimes c)}$$

for $\overline{(a, b)}, \overline{(c, d)} \in \mathbb{S}$. The system $(\mathbb{S}, \oplus, \otimes)$ is a semiring, because $(\mathbb{S}, \oplus)$ is associative, $(\mathbb{S}, \otimes)$ is associative, and $(\mathbb{S}, \oplus, \otimes)$ satisfies both the left and right distributive.

Lemma 2.2 [2]**.** *Let* $(\mathbb{S}, \oplus, \otimes)$ *be the symmetrized max plus algebra. Then the following statements hold*:

(1) $(\mathbb{S}, \oplus, \otimes)$ *is commutative*,

(2) *an element* $\overline{(\varepsilon, \varepsilon)}$ *is a null element and an absorbent element*,

(3) *an element* $\overline{(e, \varepsilon)}$ *is a unit element*,

(4) $(\mathbb{S}, \oplus, \otimes)$ *is an additively idempotent*.

The system $\mathbb{S}$ is divided into three classes, they are:

(1) $\mathbb{S}^{\oplus} = \{\overline{(t, \varepsilon)} | t \in \mathbb{R}_{\varepsilon}\}$ with $\overline{(t, \varepsilon)} = \{(t, x) \in \mathbb{R}_{\varepsilon}^2 | x < t\}$,

(2) $\mathbb{S}^{\ominus} = \{\overline{(\varepsilon, t)} | t \in \mathbb{R}_{\varepsilon}\}$ with $\overline{(\varepsilon, t)} = \{(x, t) \in \mathbb{R}_{\varepsilon}^2 | x < t\}$,

(3) $\mathbb{S}^{\bullet} = \{\overline{(t, t)} | t \in \mathbb{R}_{\varepsilon}\}$ with $\overline{(t, t)} = \{(t, t) \in \mathbb{R}_{\varepsilon}^2\}$.

Because $\mathbb{S}^{\oplus}$ isomorphic with $\mathbb{R}_{\varepsilon}$, so it will be shown that for $a \in \mathbb{R}_{\varepsilon}$, can be expressed by $\overline{(a, \varepsilon)} \in \mathbb{S}^{\oplus}$. Furthermore, we have:

(1) $a = \overline{(a, \varepsilon)}$ with $\overline{(a, \varepsilon)} \in \mathbb{S}^{\oplus}$,

(2) $\ominus a = \ominus\overline{(a, \varepsilon)} = \overline{\ominus(a, \varepsilon)} = \overline{(\varepsilon, a)}$ with $\overline{(\varepsilon, a)} \in \mathbb{S}^{\ominus}$,

(3) $a^{\bullet} = a \ominus a = \overline{(a, \varepsilon)} \ominus \overline{(a, \varepsilon)} = \overline{(a, \varepsilon)} \oplus \overline{(\varepsilon, a)} = \overline{(a, a)} \in \mathbb{S}^{\bullet}$.

Let $\mathbb{S}$ be the symmetrized max plus algebra, a positive integer n and $M_n(\mathbb{S})$ be the set of all $n \times n$ matrices over $\mathbb{S}$. The $n \times n$ zero matrix over $\mathbb{S}$ is ε_n with $(\varepsilon_n)_{ij} = \varepsilon$ and an $n \times n$ identity matrix over $\mathbb{S}$ is E_n with $[E_n]_{ij} = \begin{cases} e, \text{ if } i = j, \\ \varepsilon, \text{ if } i \neq j. \end{cases}$ The properties of balance relation, i.e., the operator ∇, are given in the following lemma.

Lemma 2.3 [1, 2]**.** (1) $\forall a, b, c \in \mathbb{S}, a \ominus c \nabla b \Leftrightarrow a \nabla b \oplus c$,

(2) $\forall a, b \in \mathbb{S}^{\oplus} \cup \mathbb{S}^{\ominus}, a \nabla b \Rightarrow a = b$,

(3) *Let* $A \in M_n(\mathbb{S})$. *The homogeneous linear balance systems* $A \otimes x \nabla \varepsilon_{n \times 1}$ *has a nontrivial solution in* $\mathbb{S}^{\oplus}$ *or* $\mathbb{S}^{\ominus}$ *if and only if* $det(A) \nabla \varepsilon$.

3. The Main Result

In this section, we indicate how a technique that is used to obtain the necessary and sufficient condition for an existence of a general solution of a non homogeneous linear balance systems for matrix X that satisfies $A \otimes X \otimes A \nabla A$. It will be shown how to construct the set of all matrices X such that $A \otimes X \otimes A \nabla A$. The construction of the matrix X such that $A \otimes X \otimes A \nabla A$ for an arbitrary $A \in M_{m \times n}(\mathbb{S})$ is simplified by transforming A into a sequence elementary row and column operations, as shown in the following theorem. The following theorems establish the existence of the matrix X such that $A \otimes X \otimes A \nabla A$ and its applications in solving equations.

Theorem 3.1. *Let $A \in M_{m \times n}(\mathbb{S})$ with $rank_{\oplus}(A) = r$. An $n \times m$ matrix X satisfies $A \otimes X \otimes A \nabla A$ if and only if*

(1)

$$X \nabla Q \otimes \begin{pmatrix} E_r & \varepsilon \\ \varepsilon & D \end{pmatrix} \otimes P$$

for $D \in M_{(n-r) \times (m-r)}(\mathbb{S})$, $P \in M_{m \times m}(\mathbb{S})$ and $Q \in M_{n \times n}(\mathbb{S})$ with P, Q are product of the elementary matrices that satisfy

(2)

$$P \otimes A \otimes Q \nabla \begin{pmatrix} E_r & \varepsilon \\ \varepsilon & \varepsilon \end{pmatrix}.$$

Proof. ($\Leftarrow$) Rewriting (2) as

$$A \nabla P^{\otimes -1} \otimes \begin{pmatrix} E_r & \varepsilon \\ \varepsilon & \varepsilon \end{pmatrix} \otimes Q^{\otimes -1}$$

it is easily verified that any X given by (1) satisfies

$$A \otimes X \otimes A \nabla P^{\otimes -1} \otimes \begin{pmatrix} E_r & \varepsilon \\ \varepsilon & \varepsilon \end{pmatrix} \otimes Q^{\otimes -1} \otimes Q \begin{pmatrix} E_r & \varepsilon \\ \varepsilon & D \end{pmatrix}$$

$$\otimes P \otimes P^{\otimes -1} \otimes \begin{pmatrix} E_r & \varepsilon \\ \varepsilon & \varepsilon \end{pmatrix} \otimes Q^{\otimes -1}.$$

Hence, $A \otimes X \otimes A \nabla P^{\otimes -1} \otimes \begin{pmatrix} E_r & \varepsilon \\ \varepsilon & \varepsilon \end{pmatrix} \otimes Q^{\otimes -1} \nabla A.$ $(\Rightarrow)$ Let $A \otimes X \otimes A \nabla A.$ Then, both $A \otimes X$ and $X \otimes A$ satisfy

$$A \otimes X \otimes A \otimes X \nabla A \otimes X \text{ and } X \otimes A \otimes X \otimes A \nabla X \otimes A.$$

$A \otimes X$ and $X \otimes A$ have the same rank as A. Thus, both $A \otimes X$ and $X \otimes A$ are of the form $\begin{pmatrix} E_r & \varepsilon \\ \varepsilon & \varepsilon \end{pmatrix}.$ Therefore, there exists nonsingular R such that

$$R^{-1} \otimes A \otimes X \otimes R \nabla \begin{pmatrix} E_r & \varepsilon \\ \varepsilon & \varepsilon \end{pmatrix} \text{ and } Q^{-1} \otimes X \otimes A \otimes Q \nabla \begin{pmatrix} E_r & \varepsilon \\ \varepsilon & \varepsilon \end{pmatrix}.$$

Thus, $R^{-1} \otimes A \otimes Q \nabla R^{-1} \otimes A \otimes X \otimes A \otimes X \otimes A \otimes Q.$ Hence,

$$R^{-1} \otimes A \otimes Q \nabla \begin{pmatrix} E_r & \varepsilon \\ \varepsilon & \varepsilon \end{pmatrix} \otimes R^{-1} \otimes A \otimes Q \otimes \begin{pmatrix} E_r & \varepsilon \\ \varepsilon & \varepsilon \end{pmatrix}.$$

It follows that $R^{-1} \otimes A \otimes Q$ is of the form

$$R^{-1} \otimes A \otimes Q \nabla \begin{pmatrix} C & \varepsilon \\ \varepsilon & \varepsilon \end{pmatrix} \Leftrightarrow A \nabla R \otimes \begin{pmatrix} C & \varepsilon \\ \varepsilon & \varepsilon \end{pmatrix} \otimes Q^{-1}$$

with $rank_{\oplus}(C) = rank_{\oplus}(A).$ Let $P = \begin{pmatrix} C^{-1} & \varepsilon \\ \varepsilon & \varepsilon \end{pmatrix} \otimes R^{-1}.$ Then

$$P \otimes A \otimes Q \nabla \begin{pmatrix} C^{-1} & \varepsilon \\ \varepsilon & \varepsilon \end{pmatrix} \otimes R^{-1} \otimes R \otimes \begin{pmatrix} C & \varepsilon \\ \varepsilon & \varepsilon \end{pmatrix} \otimes Q^{-1} \otimes Q \nabla \begin{pmatrix} E_r & C \\ \varepsilon & \varepsilon \end{pmatrix}.$$

Consider the matrix $Q^{-1} \otimes X \otimes P^{-1}.$ We have

$$\begin{pmatrix} E_r & \varepsilon \\ \varepsilon & \varepsilon \end{pmatrix} \otimes Q^{-1} \otimes X \otimes P^{-1} \nabla P \otimes A \otimes Q \otimes Q^{-1} \otimes X \otimes P^{-1}.$$

So, $\begin{pmatrix} E_r & \varepsilon \\ \varepsilon & \varepsilon \end{pmatrix} \otimes Q^{-1} \otimes X \otimes P^{-1} \nabla P \otimes A \otimes X \otimes P^{-1} \nabla \begin{pmatrix} E_r & \varepsilon \\ \varepsilon & \varepsilon \end{pmatrix}.$ Furthermore, we have

$$Q^{-1} \otimes X \otimes P^{-1} \otimes \begin{pmatrix} E_r & \varepsilon \\ \varepsilon & \varepsilon \end{pmatrix} \nabla Q^{-1} \otimes X \otimes P^{-1} \otimes P \otimes A \otimes Q.$$

Consequently,

$$Q^{-1} \otimes X \otimes P^{-1} \otimes \begin{pmatrix} E_r & \varepsilon \\ \varepsilon & \varepsilon \end{pmatrix} \nabla Q^{-1} \otimes X \otimes A \otimes Q \nabla \begin{pmatrix} E_r & \varepsilon \\ \varepsilon & \varepsilon \end{pmatrix}.$$

We conclude from the previous forms, that is

$$Q^{-1} \otimes X \otimes P^{-1} \nabla \begin{pmatrix} E_r & \varepsilon \\ \varepsilon & D \end{pmatrix}$$

for arbitrary D. Finally, $X \nabla Q \otimes \begin{pmatrix} E_r & \varepsilon \\ \varepsilon & D \end{pmatrix} \otimes P$. This completes the proof. □

According to Theorem 3.1, we give the following example:

Example 3.2. Let $A = \begin{pmatrix} \ominus 2 & 1^{\bullet} & \varepsilon & 0 \\ \varepsilon & 1 & \varepsilon & \ominus 0 \\ 1 & 0^{\bullet} & 1 & \varepsilon \end{pmatrix}$.

We have $P \otimes A \otimes Q = \begin{pmatrix} e & (-1)^{\bullet} & \varepsilon & (-2)^{\bullet} \\ \varepsilon & e & \varepsilon & (-1)^{\bullet} \\ 0^{\bullet} & (-1)^{\bullet} & \varepsilon & (-2)^{\bullet} \end{pmatrix} \nabla (E_3 \ \varepsilon)$ with $P = E_{3(-1)} \otimes E_{32(0)} \otimes E_{2(-1)} \otimes E_{31(\ominus 1)} \otimes E_{1(\ominus(-2))}$

$$P = \begin{pmatrix} e & \varepsilon & \varepsilon \\ \varepsilon & e & \varepsilon \\ \varepsilon & \varepsilon & -1 \end{pmatrix} \otimes \begin{pmatrix} e & \varepsilon & \varepsilon \\ \varepsilon & e & \varepsilon \\ \varepsilon & e & e \end{pmatrix} \otimes \begin{pmatrix} e & \varepsilon & \varepsilon \\ \varepsilon & -1 & \varepsilon \\ \varepsilon & \varepsilon & e \end{pmatrix} \otimes \begin{pmatrix} e & \varepsilon & \varepsilon \\ \varepsilon & e & \varepsilon \\ \ominus 1 & \varepsilon & e \end{pmatrix}$$

$$\otimes \begin{pmatrix} \ominus(-2) & \varepsilon & \varepsilon \\ \varepsilon & e & \varepsilon \\ \varepsilon & \varepsilon & e \end{pmatrix} = \begin{pmatrix} \ominus(-2) & \varepsilon & \varepsilon \\ \varepsilon & -1 & \varepsilon \\ -2 & -2 & -1 \end{pmatrix}$$

and

$$Q = E_{42(-1)} \otimes E_{41(-2)} = \begin{pmatrix} 0 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 0 & \varepsilon & -1 \\ \varepsilon & \varepsilon & 0 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & \varepsilon & \varepsilon & -2 \\ \varepsilon & 0 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \varepsilon & \varepsilon & -2 \\ \varepsilon & 0 & \varepsilon & -1 \\ \varepsilon & \varepsilon & 0 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 0 \end{pmatrix}.$$

There is

$$X \nabla Q \otimes \begin{pmatrix} E_3 \\ \varepsilon \end{pmatrix} \otimes P \nabla Q \otimes \begin{pmatrix} e & \varepsilon & \varepsilon \\ \varepsilon & e & \varepsilon \\ \varepsilon & \varepsilon & e \\ \varepsilon & \varepsilon & \varepsilon \end{pmatrix} \otimes P \nabla \begin{pmatrix} \ominus(-2) & \varepsilon & \varepsilon \\ \varepsilon & -1 & \varepsilon \\ -2 & -2 & -1 \\ \varepsilon & \varepsilon & \varepsilon \end{pmatrix}$$

satisfies $A \otimes X \otimes A \nabla A$.

Theorem 3.3. *Let* $A \in M_{m \times n}(\mathbb{S})$. *If* X *is any matrix satisfying* $A \otimes X \otimes A \nabla A$, *then* $A \otimes x \nabla b$ *has a solution if and only if* $A \otimes X \otimes b \nabla b$, *in which case the most general solution is* $x = X \otimes b \oplus (E \ominus X \otimes A) \otimes h$, *where h is arbitrary*.

Proof.

$$A \otimes x = A \otimes [X \otimes b \oplus (E \ominus X \otimes A) \otimes h]$$

$$= A \otimes X \otimes b \oplus A \otimes (E \ominus X \otimes A) \otimes h$$

$$= (A \otimes X \otimes b) \oplus A \otimes h \ominus (A \otimes X \otimes A) \otimes h \nabla b \oplus A$$

$$\otimes h \ominus A \otimes h \nabla b \oplus (A \otimes h)^{\bullet}.$$

Because we have $(A \otimes h)^{\bullet} \nabla \varepsilon$, we conclude that $A \otimes x \nabla b$. □

Corollary 3.4. *If X is any matrix satisfying $A \otimes X \otimes A \nabla A$, then $A \otimes x \nabla \varepsilon$ has a solution if and only if the most general solution is $x = (E \ominus X \otimes A) \otimes h$, where h is arbitrary.*

Proof. $A \otimes x = A \otimes (E \ominus X \otimes A) \otimes h = A \otimes h \ominus A \otimes X \otimes A \otimes h = A \otimes h \ominus A \otimes h$. Because $A \otimes h \ominus A \otimes h = (A \otimes h)^{\bullet}$ and $(A \otimes h)^{\bullet} \nabla \varepsilon$, we conclude that $A \otimes x \nabla \varepsilon$. □

Corollary 3.5. *Vector $x \nabla \begin{pmatrix} \ominus C \otimes y \\ y \end{pmatrix}$, where y is arbitrary, is the general solution from the linear balance systems $A \otimes x \nabla \varepsilon$, if and only if X that has $\begin{pmatrix} E_r & \varepsilon \\ \varepsilon & D \end{pmatrix} \otimes P$ form where D is arbitrary, is any matrix satisfying $A \otimes X \otimes A \nabla A$, which $A \nabla P^{\otimes -1} \otimes \begin{pmatrix} E_r & C \\ \varepsilon & \varepsilon \end{pmatrix}$.*

Proof. According to Corollary 3.4, we have

$$x = (E \ominus X \otimes A) \otimes h \nabla \left[E \ominus \left(\begin{pmatrix} E_r & \varepsilon \\ \varepsilon & D \end{pmatrix} \otimes P \right) \right.$$

$$\left. \otimes \left(P^{\otimes -1} \otimes \begin{pmatrix} E_r & C \\ \varepsilon & \varepsilon \end{pmatrix} \right) \right] \otimes h.$$

Furthermore, we obtain

$$x \nabla \left[E \ominus \begin{pmatrix} E_r & \varepsilon \\ \varepsilon & D \end{pmatrix} \otimes \begin{pmatrix} E_r & C \\ \varepsilon & \varepsilon \end{pmatrix} \right] \otimes h \nabla \left[E \ominus \begin{pmatrix} E_r & C \\ \varepsilon & \varepsilon \end{pmatrix} \right] \otimes h.$$

If we take $E = \begin{pmatrix} E_r & \varepsilon \\ \varepsilon & E_{m-r} \end{pmatrix}$ and $h = \begin{pmatrix} h_r \\ h_{m-r} \end{pmatrix}$, then we obtain that x can be presented as the following form:

$$x \nabla \left[\begin{pmatrix} E_r & \varepsilon \\ \varepsilon & E_{m-r} \end{pmatrix} \ominus \begin{pmatrix} E_r & C \\ \varepsilon & \varepsilon \end{pmatrix} \right] \otimes \begin{pmatrix} h_r \\ h_{m-r} \end{pmatrix}.$$

Hence, $x \nabla \begin{pmatrix} \varepsilon & \ominus C \\ \varepsilon & E_{m-r} \end{pmatrix} \otimes \begin{pmatrix} h_r \\ h_{m-r} \end{pmatrix} \nabla \begin{pmatrix} \ominus C \otimes h_{m-r} \\ h_{m-r} \end{pmatrix}$. We now conclude that $x \nabla \begin{pmatrix} \ominus C \otimes y \\ y \end{pmatrix}$, where y is arbitrary, is the solution of $A \otimes x \nabla \varepsilon$. □

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