A Note of The Linear Equation AX = B with Multiplicatively-Reguler Matrix A in Semiring

by Gregoria Ariyanti

A Note of The Linear Equation $AX = B$ with Multiplicatively-Reguler Matrix A in Semiring

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Abstract. Semiring is a form of generalization of the ring, where one or more conditions in the ring are removed. An element a is called multiplicatively-regular if there is x so $axa = a$. In real number algebra, a system of linear equations $AX = B$ has a singular solution if a matrix A has an inverse. Elements of semiring which does not a zero element have no inverse of addition. By reviewing matrix A as a multiplicatively-regular, it is develop of necessary or sufficient condition of semiring. Given a matrix A with the right complement matrix A^r satisfies $AA^r = 0$. The sufficient condition of the linear equations system $AX = B$ has a solution is there exist a matrix B satisfies $AA^{\circ}B = B$ and a matrix A has a right complement matrix A^r.

1. Introduction

A non-empty set G with a binary operation $*$ is called Group if it has the following properties: associative, has an identity element of binary operations $*$, and every element that is not an identity element has an inverse. Meanwhile, a non-empty set R with two binary operations namely $*$ and $\circ$ is called Ring if it has the following properties: $(R, *)$ is a commutative group, closed to binary operation $\circ$, associative to binary operation $\circ$, and to both binary operations $*$ and $\circ$ is distributive. If the ring has the following properties: commutative to binary operation $\circ$, it has a unit element of binary operation $\circ$, and every element that is not a zero element has an inverse to binary operation $\circ$, it is called Field. If the conditions of a Group and Ring are weakened, other algebraic structures will appear, namely Semigrup and Semiring. That is, if some Group or Ring conditions are removed, the algebraic structure formed is Semigrup and then Semiring [2,6]. One of the problems and applications that are often encountered in mathematics is to complete the Linear Equations System [2,5].

The linear equations system that has been developed by researchers is a system of linear equations over Field which include real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$ [5]. In other studies, the object of research is extended not to Field anymore, but to commutative Ring and linear equations system Commutative over ring have been discussed by Brewer, et al. [3]. Likewise, assuming an extension from the Ring to Ring commutative does not change the definition in general [4].

In this paper, we show about matrix in linear equations system over semiring. In Section 2, we will review some basic facts for semiring, matrix over semiring, and the linear equations system over semiring. In Section 3, we show the results that is the necessary or sufficient condition of

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solution of linear equations system over semiring by reviewing A as a multiplicatively-regular and a right complement matrix.

2. Some Preliminaries on Semiring

In this section, we review some basic facts for the semiring and a matrix over the semiring.

2.1. *Semiring*

Definition 2.1. ([6]) *Semigroup S is an empty set that is equipped with an associative binary $*$ operation, that is for every $x, y, z \in S, x * (y * z) = (x * y) * z$.*

Furthermore, Poplin provides the following definition.

Definition 2.2. ([6]) *Semiring is a non-empty set S has two binary operations, addition $(+)$ and multiplication $(\times)$, which has the following properties.*

(i) Operation $+$ is commutative and associative, that is
   (a) $a + b = b + a$ for every $a, b \in S$,
   (b) $(a + b) + c = a + (b + c)$ for each $a, b, c \in S$.

(ii) Operation $\times$ is associative and distributive to $+$, i.e.
   (a) $(ab)c = a(bc)$ for each $a, b, c \in S$
   (b) $a(b + c) = (ab) + (ac)$ for each $a, b, c \in S$,
   (c) $(b + c)a = (ba) + (ca)$ for each $a, b, c \in S$.

(iii) The set S has a zero element $0 \in S$ so that
   (a) $0 + a = a + 0 = a$ for every $a \in S$
   (b) $0 \times a = a \times 0 = 0$ for each $a \in S$ and here in after called the absorbent element (absorption).

(iv) The set S has a unit element e, $e \times a = a \times e = a$ for every $a \in S$.

As in the Group and Ring structure, the commutative and idempotent nature also applies to certain Semiring. This is as stated by Poplin ([6]) below.

Poplin [6] stated if $\times$ operation is commutative then S is called a commutative semiring and if $+$ operation is idempoten then S is called a idempotent semiring.

2.2. *Matrices over Semiring*

We let $M_{n\times 1}(S)$ is the set of all $n \times 1$ vectors with elements from semiring S. We also let $M_{n\times n}(S)$ is the set of all $n \times n$ matrices with elements from semiring S. The $+$ and $\times$ operation for a matrices over semiring is defined as :

Definition 2.3. *Let S semiring, a positive integer n and $M_n(S)$ is the set of all $n \times n$ matrices over $\mathbb{S}$. For every $A, B \in M_n(S)$, $+$ and $\times$ operations over semiring S are defined :*

$$C = A + B \quad \Rightarrow \quad c_{ij} = a_{ij} + b_{ij}$$
$$C = A \times B \quad \Rightarrow \quad c_{ij} = \sum_{l} a_{il} \times b_{lj}$$

Therefore semiring has 0 as a zero element and 1 as an identity element, as in the matrix of conventional algebra, then a zero matrix and a identity matrix can be formed. The zero matrix $n \times n$ over semiring S is 0_n is defined as matrix with all elements equal to the $0-$element, that is $(0_n)_{ij} = 0$. The identity matrix $n \times n$ over S is defined as the matrix with all elements equal to the $e-$element, that is I_n with $[I_n]_{ij} = \begin{cases} e, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$.

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2.3. The Linear Equations System over Semiring

We consider linear equations system over semirings S whose equations and variables are indexed by arbitrary sets, not necessarily ordered. In the following, if I, J and S are finite and non-empty sets then an $I \times J$ matrix over S is a function $A : I \times J \to S$. An $I-$vector over S is defined similarly as a function $b : I \to S$. We write (A, b) as a matrix equation $A \cdot x = b$, where x is a $J-$vector of variables in S. The system (A, b) is said to be solvable if there exists a solution vector $c : J \to S$ such that $A \cdot c = b$, where we define multiplication of unordered matrices and vectors in the usual way by $(A \cdot c)(i) = \sum_{j \in J} A(i, j) \cdot c(j)$ for all $i \in I$. Similarly, a system of linear equations over a commutative semiring S is a pair (A, b) where A is an $I \times J$ matrix with entries in S and b is an $I-$vector over S. The linear equations system can be expressed in other forms, as in the following description. We consider linear equations system

$$\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \end{array} \tag{1}$$

The linear equations system (1) represented m linear equations with n arbitrary variable.

The matrix for the system of equations above are

$$Ax = B \tag{2}$$

with matrix $A = (a_{ij}) \in M_{m \times n}(S)$, $B = \begin{pmatrix} b_1 & b_2 & \ldots & b_m \end{pmatrix}^T \in S^m$, and $x = \begin{pmatrix} x_1 & x_2 & \ldots & x_m \end{pmatrix}^T \in S^m$ ([1]).

The equation in (1) or (2) stated had a solution in S^n, if there a vector $\zeta \in S^n$ such that $A\zeta = B$. If $B = 0$ then the linear equations system $Ax = 0$ is called homogeneous system. The homogeneous system has at least one solution, that is $\zeta = 0 = \begin{pmatrix} 0 & 0 & \ldots & 0 \end{pmatrix}^T \in S^m$. The solution $\zeta = 0$ is called a trivial solution in the linear equations system $Ax = 0$. Furthermore, a vector $\zeta \in S^n$ is called a non trivial solution in $Ax = 0$ if $\zeta \neq 0$ and $A\zeta = 0$ ([5]).

3. The Results

3.1. Properties of Elements of Matrices Over Semiring

Definition 3.1. *Given a semiring S. An a in S is called multiplicatively-reguler element if and only if there exist a^* in S such that $aa^*a = a$.*

Let $a^\circ = a^*aa^*$, we have

$$aa^\circ a = aa^*aa^*a = aa^*a = a$$

and

$$a^\circ aa^\circ = a^*aa^*aa^*aa^* = a^*aa^*aa^* = a^*aa^*a = a^\circ \tag{3}$$

From above definition, we can describe the following properties.

Lemma 3.2. *Let S semiring and $a \in S$. If a is multiplicatively-reguler element then $a^\circ, a^\circ a$, and aa° are multiplicatively-reguler element.*

Proof. Let a is a multiplicatively-reguler element and $a^\circ = a^*aa^*$. We have the following results:

(i) $a^\circ a^* a^\circ = a^*aa^*a^*a^*aa^* = a^*aa^*aa^* = a^*aa^* = a^\circ$.

(ii) $(a^\circ a)a^*(a^\circ a) = (a^*aa^*a)a^*(a^*aa^*a) = a^*aa^*a^*a = a^*aa^*a = a^\circ a$.

(iii) $(aa^\circ)a^*(aa^\circ) = aa^*aa^*a^*aa^*aa^* = aa^*aa^*aa^*aa^* = aa^*aa^\circ = aa^\circ$.

Therefore, $a^\circ, a^\circ a$, and aa° are multiplicatively-reguler elements. □

From the set of all multiplicatively-reguler elements of S, we can describe two functions that given relations from S to the set $F(S)$ of all multiplicatively element. The set $F(S)$ is formed all multiplicatively element of S given two function, that are λ and ϕ. The λ function is a relation from a to $a^\circ a$ and the ϕ function is a relation from a to aa°, and those functions satisfy $\lambda^2 = \lambda$ and $\phi^2 = \phi$. Those functions are idempotent.

Lemma 3.3. *For a in semiring S, we have the following properties*

(i) $a\lambda(a) = aa^\circ a = a = \phi(a)a$

(ii) $\lambda(a)a^\circ = a^\circ aa^\circ = a^\circ = a^\circ\phi(a)$

Proof. Let S is semiring and $a \in S$. We have the following results:

(i) $a\lambda(a) = aa^\circ a = aa^*aa^*a = aa^*a = a$ and $a\lambda(a) = aa^\circ a = \phi(a)a$.

(ii) $\lambda(a)a^\circ = (a^\circ a)a^\circ = a^*aa^*aa^*aa^* = a^\circ aa^\circ = a^\circ$ and $\lambda(a)a^\circ = a^\circ aa^\circ = a^\circ\phi(a)$.

□

Theorem 3.4. *Let S is semiring and a, b are multiplicatively-reguler elements in S. The form $ax = b$ has a solution if and only if b satisfies $\phi(a)b = b$.*

Proof. ($\Rightarrow$) Let $ax = b$ has a solution, that is $a^\circ b = x$. We have, $\phi(a)b = \phi(a)ax = aa^o ax = ax = b$. Therefore, we have b that satisfies $\phi(a)b = b$.
($\Leftarrow$) Let b satisfies $\phi(a)b = b$. We have, $ax = a(a^\circ b) = \phi(a)b = b$. It means that $ax = b$ has a solution, that is $a^\circ b = x$ □

Definition 3.5. *Let a is element in semiring S. The form a^r is called right complement if that element satisfies $aa^r = 0$ and $a + a^r = 1$. Also, the form a^l is called left complement if that element satisfies $a^l a = 0$ and $a^l + a = 1$.*

From that definition, we have the following property.

Lemma 3.6. *Let S is semiring. If $a \in S$ has both right complement a^r and left complement a^l then $a^r = a^l$.*

Proof. We have

$$\begin{aligned} a^l &= a^l 1 = a^l(a + a^r) = a^l a + a^l a^r = a^l a^r \\ &= 0 + a^l a^r = aa^r + a^l a^r = (a + a^l)a^r = a^r. \end{aligned}$$

□

Theorem 3.7. *If a is multiplicatively-reguler element of semiring S that satisfies $\lambda(a)\lambda(a)^r = 0$, $\lambda(a) + \lambda(a)^r = 1$, and $b \in S$ satisfies $\phi(a)b = b$, then solution of $ax = b$ is $\phi_b(y) = a^\circ b + \lambda(a)^r y$.*

Proof. We have

$$\begin{aligned} a\phi_b(y) &= a(a^\circ b + \lambda(a)^r y) \\ &= aa^\circ b + a\lambda(a)^r y \\ &= \phi(a)b + a\lambda(a)\lambda(a)^r y \\ &= \phi(a)b, \text{with} \lambda(a)\lambda(a)^r = 0 \\ &= b \end{aligned}$$

□

3.2. ***Existence of Solution of The Equations System with The Multiplicatively-Reguler Matrix***

To develop the characteristics of the matrix such that the equation linear system $AX = B$ has solution, it is carried through explores the matrix by referring to the properties of the multiplicatively-reguler elements and complement elements in semiring S. Let S is semiring and $M_{n\times n}(S)$ is matrix over semiring S. The multiplicatively-reguler matrix A is A^* which satisfies $AA^*A = A$. If there exist a multiplicatively-reguler matrix then $A^\circ = A^*AA^*$, so we have $AA^\circ A = AA^*AA^*A = AA^*A = A$.

Theorem 3.8. *Let A and B are matrices over semiring S and A is multiplicatively-reguler matrix that satisfies $AA^r = 0$ and $AA^\circ B = B$, then the linear equation $AX = B$ has solution $X = A^\circ B + A^r A$.*

Proof. Let S is semiring. We have A is multiplicatively-reguler matrix that satisfies $AA^r = 0$ and $AA^\circ B = B$. Furthermore, we have the following form.

$$\begin{aligned} AX &= A(A^\circ B + A^r A) \\ &= AA^\circ B + AA^r A \\ &= B + 0 = B \end{aligned}$$

Therefore, $X = A^\circ B + A^r A$ is solution of $AX = B$ with A and B are matrices over semiring S. □

4. Conclusion

Through explores the matrix by referring to the properties of the multiplicatively-reguler elements and complement elements in semiring S, we develop the characteristics of the matrix such that the equation linear system $AX = B$ has solution. Let S is semiring and $M_{n\times n}(S)$ is matrix over semiring S. The multiplicatively-reguler matrix A is A^* which satisfies $AA^*A = A$. If there exist a multiplicatively-reguler matrix then $A^\circ = A^*AA^*$, so we have $AA^\circ A = AA^*AA^*A = AA^*A = A$. From above characteristic, we get clonclusion, that is, for A and B are matrices over semiring S and A is multiplicatively-reguler matrix that satisfies $AA^r = 0$ and $AA^\circ B = B$, then the linear equation $AX = B$ has solution $X = A^\circ B + A^r A$.

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