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NECESSARY AND SUFFICIENT CONDITIONS FOR THE SOLUTION OF THE LINEAR BALANCED SYSTEMS IN THE SYMMETRIZED MAX PLUS ALGEBRA

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Abstract

The system max plus algebra does not have an additive inverse. Therefore, some equations do not have a solution. For example, the equation $3 \oplus x = 2$ has no solution since there is no x such that $\max(3, x) = 2$. One way of trying to solve this problem is to extend the max plus algebra to a larger system which will include additive inverse in the same way that the natural numbers were extended to the larger system of integers. The extended system that is larger than max

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plus means the symmetrized max plus algebra. This paper describes the necessary and sufficient condition for the solution of the balanced linear system, that is, the linear system over the symmetrized max plus algebra.

It is shown that vector $x \nabla \begin{pmatrix} \ominus C \otimes y \\ y \end{pmatrix}$, where y is arbitrary, is the general solution from linear balanced system of $A \otimes x \nabla \varepsilon$ if and only if X that has $\begin{pmatrix} E_r & \varepsilon \\ \varepsilon & D \end{pmatrix} \otimes P$ form where D is arbitrary, is any matrix satisfying $A \otimes X \otimes A \nabla A$, which $A \nabla P^{\otimes^{-1}} \otimes \begin{pmatrix} E_r & C \\ \varepsilon & \varepsilon \end{pmatrix}$.

1. Introduction

The system max plus algebra does not have an additive inverse. Therefore, some equations do not have a solution. For example, the equation $3 \oplus x = 2$ has no solution since there is no x such that $\max(3, x) = 2$. De Schutter and De Moor [2] and Singh et al. [5] stated that one way of trying to solve this problem is to extend the max plus algebra to a larger system which will include additive inverse in the same way that the natural numbers were extended to the larger system of integers. Therefore, we have that the system $(\mathbb{S}, \oplus, \otimes)$ is called the *symmetrized max plus algebra* and $\mathbb{S} = \mathbb{R}_\varepsilon^2 / \mathcal{B}$ with $\mathcal{B}$ is an equivalence relation. Malešević et al. [4] stated about the solution of a linear system $Ax = c$. This paper describes the necessary and sufficient condition for the solution of the balanced linear system, that is, the linear system over the symmetrized max plus algebra.

1.1. The symmetrized max plus algebra

Let the set of all real numbers $\mathbb{R}_\varepsilon = \mathbb{R} \bigcup \{\varepsilon\}$ with $\varepsilon := -\infty$ as the null element and $e := 0$ as the unit element. For all $a, b \in \mathbb{R}_\varepsilon$, the operations $\oplus$ and $\otimes$ are defined as follows:

$$a \oplus b = \max(a, b) \quad \text{and} \quad a \otimes b = a + b$$

and then, $(\mathbb{R}_\varepsilon, \oplus, \otimes)$ is called the *max plus algebra*.

Definition 1.1 (De Schutter and De Moor [2] and Kondo [3])**.** Let $u = (x,\ y),\ \ v = (w,\ z) \in \mathbb{R}_{\varepsilon}^{2}.$

(a) Two unary operators $\ominus$ and $(\cdot)^{\bullet}$ are defined as follows:

$$\ominus u = (y,\ x) \quad \text{and} \quad u^{\bullet} = u \oplus (\ominus u).$$

(b) An element *u* is called *balances* with *v*, denoted by $u \nabla v$, if

$$x \oplus z = y \oplus w.$$

(c) A relation $\mathcal{B}$ is defined as follows:

$$(x,\ y)\mathcal{B}(w,\ z) \text{ if } \begin{cases} (x,\ y)\nabla(w,\ z), & \text{if } x \neq y \text{ and } w \neq z, \\ (x,\ y) = (w,\ z), & \text{otherwise.} \end{cases}$$

Because $\mathcal{B}$ is an equivalence relation, we have the set of factors $\mathbb{S} = \mathbb{R}_{\varepsilon}^{2}/\mathcal{B}$ and the system $(\mathbb{S},\ \oplus,\ \otimes)$ is called the *symmetrized max plus algebra*, with the operations of addition and multiplication on $\mathbb{S}$ is defined as follows:

$$\overline{(a,\ b)} \oplus \overline{(c,\ d)} = \overline{(a \oplus c,\ b \oplus d)},$$

$$\overline{(a,\ b)} \oplus \overline{(c,\ d)} = \overline{(a \otimes c \oplus b \otimes d,\ a \otimes d \oplus b \otimes c)}$$

for $\overline{(a,\ b)},\ \overline{(c,\ d)} \in \mathbb{S}.$

The system $(\mathbb{S},\ \oplus,\ \otimes)$ is a semiring, because $(\mathbb{S},\ \oplus)$ is associative, $(\mathbb{S},\ \otimes)$ is associative, and $(\mathbb{S},\ \oplus,\ \otimes)$ satisfies both the left and right distributive.

Lemma 1.2 (De Schutter and De Moor [2])**.** *Let* $(\mathbb{S},\ \oplus,\ \otimes)$ *be the symmetrized max plus algebra*. *Then the following statements hold*:

(1) $(\mathbb{S},\ \oplus,\ \otimes)$ *is commutative*.

(2) *An element* $\overline{(\varepsilon,\ \varepsilon)}$ *is a null element and an absorbent element*.

(3) *An element* $\overline{(e,\ \varepsilon)}$ *is a unit element*.

(4) $(\mathbb{S},\ \oplus,\ \otimes)$ *is an additively idempotent*.

The system $\mathbb{S}$ is divided into three classes:

- $\mathbb{S}^{\oplus}$ consists of all positive elements or

$$\mathbb{S}^{\oplus} = \{\overline{(t, \varepsilon)} \mid t \in \mathbb{R}_{\varepsilon}\} \text{ with } \overline{(t, \varepsilon)} = \{(t, x) \in \mathbb{R}_{\varepsilon}^2 \mid x < t\}.$$

- $\mathbb{S}^{\ominus}$ consists of all negative elements or

$$\mathbb{S}^{\ominus} = \{\overline{(\varepsilon, t)} \mid t \in \mathbb{R}_{\varepsilon}\} \text{ with } \overline{(\varepsilon, t)} = \{(x, t) \in \mathbb{R}_{\varepsilon}^2 \mid x < t\}.$$

- $S^{\bullet}$ consists of all balanced elements or

$$\mathbb{S}^{\bullet} = \{\overline{(t, t)} \mid t \in \mathbb{R}_{\varepsilon}\} \text{ with } \overline{(t, t)} = \{(t, t) \in \mathbb{R}_{\varepsilon}^2\}.$$

Because $\mathbb{S}^{\oplus}$ isomorphic with $\mathbb{R}_{\varepsilon}$, so it will be shown that for $a \in \mathbb{R}_{\varepsilon}$, it can be expressed by $\overline{(a, \varepsilon)} \in \mathbb{S}^{\oplus}$.

Furthermore, it is easy to verify that for $a \in \mathbb{R}_{\varepsilon}$, we have:

$$a = \overline{(a, \varepsilon)} \text{ with } \overline{(a, \varepsilon)} \in \mathbb{S}^{\oplus},$$

$$\ominus a = \ominus\overline{(a, \varepsilon)} = \overline{\ominus(a, \varepsilon)} = \overline{(\varepsilon, a)} \text{ with } \overline{(\varepsilon, a)} \in \mathbb{S}^{\ominus},$$

$$a^{\bullet} = a \ominus a = \overline{(a, \varepsilon)} \ominus \overline{(a, \varepsilon)} = \overline{(a, \varepsilon)} \oplus \overline{(\varepsilon, a)} = \overline{(a, a)} \in \mathbb{S}^{\bullet}.$$

Lemma 1.3. *For* $a, b \in \mathbb{R}_{\varepsilon}$, $a \ominus b = \overline{(a, b)}$.

Proof.

$$a \ominus b = \overline{(a, \varepsilon)} \ominus \overline{(b, \varepsilon)} = \overline{(a, \varepsilon)} \oplus \overline{(\varepsilon, b)} = \overline{(a, b)}. \qquad \square$$

Lemma 1.4. *For* $\overline{(a, b)} \in \mathbb{S}$ *with* $a, b \in \mathbb{R}_{\varepsilon}$, *the following statements hold*:

(1) *If* $a > b$, *then* $\overline{(a, b)} = \overline{(a, \varepsilon)}$.

(2) *If* $a < b$, *then* $\overline{(a, b)} = \overline{(\varepsilon, b)}$.

(3) *If* $a = b$, *then* $\overline{(a, b)} = \overline{(a, a)}$ *or* $\overline{(a, b)} = \overline{(b, b)}$.

Proof. (1) For $a > b$, we have that $a \oplus b = a$. In other words, $a \oplus \varepsilon = a \oplus b$. The result that $(a, b)\nabla(a, \varepsilon)$. So it follows that $(a, b)\mathcal{B}(a, \varepsilon)$. Therefore, $\overline{(a, b)} = \overline{(a, \varepsilon)}$.

(2) For $a < b$, we have that $a \oplus b = b$. In other words, $a \oplus b = b \oplus \varepsilon$. The result that $(a, b)\nabla(\varepsilon, b)$. So it follows that $(a, b)\mathcal{B}(\varepsilon, b)$. Therefore, $\overline{(a, b)} = \overline{(\varepsilon, b)}$. □

Corollary 1.5. *For* $a, b \in \mathbb{R}_\varepsilon$,

$$a \ominus b = \begin{cases} a, & \text{if } a > b, \\ \ominus b, & \text{if } a < b, \\ a^\bullet, & \text{if } a = b. \end{cases}$$

Let $\mathbb{S}$ be the symmetrized max plus algebra, n be a positive integer greater than 1 and $M_n(\mathbb{S})$ be the set of all $n \times n$ matrices over $\mathbb{S}$. Operations $\oplus$ and $\otimes$ for matrices over the symmetrized max plus algebra are defined:

$$C = A \oplus B \Rightarrow c_{ij} = a_{ij} \oplus b_{ij},$$

$$C = A \otimes B \Rightarrow c_{ij} = \bigoplus_l a_{il} \otimes b_{lj}.$$

Zero matrix $n \times n$ over $\mathbb{S}$ is ε_n with $(\varepsilon_n)_{ij} = \varepsilon$ and identity matrix $n \times n$ over $\mathbb{S}$ is E_n with

$$[E_n]_{ij} = \begin{cases} e, & \text{if } i = j, \\ \varepsilon, & \text{if } i \neq j. \end{cases}$$

Definition 1.6. We say that the matrix $A \in M_n(\mathbb{S})$ is *invertible* over $\mathbb{S}$ if

$$A \otimes B \nabla E_n \quad \text{and} \quad B \otimes A \nabla E_n$$

for any $B \in M_n(\mathbb{S})$.

The properties of balance relation, i.e., the operator ∇, are given by the following lemma:

Lemma 1.7 (De Schutter and De Moor [2]).

(1) $\forall a, b, c \in \mathbb{S}, a \ominus c \nabla b \Leftrightarrow a \nabla b \oplus c.$

(2) $\forall a, b \in \mathbb{S}^{\oplus} \bigcup \mathbb{S}^{\ominus}, a \nabla b \Rightarrow a = b.$

(3) *Let* $A \in M_n(\mathbb{S})$. *The homogeneous linear balance* $A \otimes x \nabla \varepsilon_{n\times 1}$ *has a non-trivial solution in* $\mathbb{S}^{\oplus}$ *or* $\mathbb{S}^{\ominus}$ *if and only if* $\det(A) \nabla \varepsilon.$

1.2. Reduced row echelon form over the symmetrized max plus algebra

Because the elements of the symmetrized max plus algebra are invertible, then it can be developed elementary row operations, that used to manipulate linear balanced systems.

Definition 1.8. The three types of elementary row operations on *A* matrix over the symmetrized max plus algebra are as follows:

(1) Interchange rows *i* and *j*.

(2) Replace row *i* by a nonzero multiple of itself $(\alpha \otimes A_i)$.

(3) Replace row *j* by a combination of itself plus a multiple of row i $(A_j \oplus \alpha \otimes A_i)$.

The new balance linear systems that are obtained from the old balance linear systems by performing a sequence of elementary row operations have the same solution.

Whenever *B* can be obtained from *A* by performing a sequence of elementary row operations only, we write $A \sim_{row} B$, and we say that *A* and *B* are *row equivalent*. In other words,

$$A \sim_{row} B \Leftrightarrow P \otimes A \nabla B$$

for a non-balanced matrix *P*.

Matrix A can be obtained by performing row operation on matrix B because every row operation is reversible. In particular, the inverse of any row operation is again a row operation of the same type.

Definition 1.9. An $m \times n$ matrix E with rows E_{i*} and columns E_{*j} is said to be in *row echelon form* provided the following two conditions hold:

(1) If E_{i*} consists entirely of ε, then all rows below E_{i*} are also entirely ε; i.e., all ε rows are at the bottom.

(2) If the first non ε entry in E_{i*} lies in the *j*th position, then all entries below the *i*th position in columns $E_{*1}, E_{*2}, \ldots, E_{*j}$ are ε.

Adding the following conditions to conditions (1) and (2) in Definition 1.9, we have a matrix E as a reduced row echelon form:

(1) The leading entry in each nonzero row is *e*.

(2) Each leading *e* is the only nonzero entry in its column.

A typical structure for a matrix in reduced row echelon form is illustrated below:

$$\begin{pmatrix} 0 & * & \varepsilon & \varepsilon & * & * & * \\ \varepsilon & \varepsilon & 0 & \varepsilon & * & * & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 0 & * & * & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 \end{pmatrix}.$$

For a matrix A, the symbol E_A denotes the unique reduced row echelon form that derived from A by means of row operations.

The following is an example of the reduced row echelon form obtained by using only row operations.

Example 1.10. Let $A = \begin{pmatrix} \ominus 2 & 1^{\bullet} & \varepsilon & 0 \\ \varepsilon & 1 & \varepsilon & \ominus 0 \\ 1 & 0^{\bullet} & 1 & \varepsilon \end{pmatrix}.$

We have a sequence of elementary row operations from A as follows:

$$\begin{pmatrix} \ominus 2 & 1^{\bullet} & \varepsilon & 0 \\ \varepsilon & 1 & \varepsilon & \ominus 0 \\ 1 & 0^{\bullet} & 1 & \varepsilon \end{pmatrix} \underset{\sim}{H_{1(\ominus(-2))}} \begin{pmatrix} 0 & (-1)^{\bullet} & \varepsilon & \ominus(-2) \\ \varepsilon & 1 & \varepsilon & \ominus 0 \\ 1 & 0^{\bullet} & 1 & \varepsilon \end{pmatrix} \underset{\sim}{H_{31(\ominus 1)}} \begin{pmatrix} 0 & (-1)^{\bullet} & \varepsilon & \ominus(-2) \\ \varepsilon & 1 & \varepsilon & \ominus 0 \\ 1^{\bullet} & 0^{\bullet} & 1 & -1 \end{pmatrix}$$

$$\underset{\sim}{H_{2(-1)}} \begin{pmatrix} 0 & (-1)^{\bullet} & \varepsilon & \ominus(-2) \\ \varepsilon & 0 & \varepsilon & \ominus(-1) \\ 1^{\bullet} & 0^{\bullet} & 1 & -1 \end{pmatrix} \underset{\sim}{H_{32(0)}} \begin{pmatrix} 0 & (-1)^{\bullet} & \varepsilon & \ominus(-2) \\ \varepsilon & 0 & \varepsilon & \ominus(-1) \\ 1^{\bullet} & 0^{\bullet} & 1 & (-1)^{\bullet} \end{pmatrix}$$

$$\underset{\sim}{H_{3(-1)}} \begin{pmatrix} 0 & (-1)^{\bullet} & \varepsilon & \ominus(-2) \\ \varepsilon & 0 & \varepsilon & \ominus(-1) \\ 0^{\bullet} & (-1)^{\bullet} & 0 & (-2)^{\bullet} \end{pmatrix}.$$

We can see that

$$\begin{pmatrix} 0 & (-1)^{\bullet} & \varepsilon & \ominus(-2) \\ \varepsilon & 0 & \varepsilon & \ominus(-1) \\ 0^{\bullet} & (-1)^{\bullet} & 0 & (-2)^{\bullet} \end{pmatrix} \nabla \begin{pmatrix} 0 & \varepsilon & \varepsilon & \star \\ \varepsilon & 0 & \varepsilon & \star \\ \varepsilon & \varepsilon & 0 & \star \end{pmatrix} = E_A.$$

2. The Main Result

In this section, we indicate how a technique is used to obtain the necessary and sufficient condition for an existence of a general solution of a non-homogeneous linear balanced system.

To show the existence of the solution of the balanced linear system uses some of the following theorems:

Theorem 2.1. *The form* $A \otimes X \otimes A \nabla A$ *has no unique solution for any* A.

If A is an $m \times n$ non-balanced matrix, then there exists an inverse $A^{\otimes(-1)}$ with the property $A \otimes A^{\otimes(-1)} \nabla A^{\otimes(-1)} \otimes A \nabla E$.

It will be shown how to construct the set of all matrices X such that $A \otimes X \otimes A \nabla A$. The construction of the matrix X such that $A \otimes X \otimes A \nabla A$

for an arbitrary $A \in M_{m\times n}(S)$ is simplified by transforming A into a sequence of elementary row operations, as shown in the following theorem. From that process, we have $A \sim_{row} A_r$, and any P such that

$$P \otimes A \nabla \begin{pmatrix} E_r & C \\ \varepsilon & \varepsilon \end{pmatrix} \tag{2.1}$$

with $C \in M_{r\times(n-r)}(\mathbb{S})$.

The following theorems establish the existence of the matrix X such that $A \otimes X \otimes A \nabla A$ and its applications in solving equations.

Theorem 2.2. *Let* $A \in M_{m\times n}(\mathbb{S})$, *and let* $P \in M_{m\times m}(\mathbb{S})$ *be such that*

$$P \otimes A \nabla \begin{pmatrix} E_r & C \\ \varepsilon & \varepsilon \end{pmatrix}. \tag{2.2}$$

Then for any $D \in M_{(n-r)\times(m-r)}(\mathbb{S})$, *the* $n \times m$ *matrix*

$$X \nabla \begin{pmatrix} E_r & \varepsilon \\ \varepsilon & D \end{pmatrix} \otimes P \tag{2.3}$$

satisfies $A \otimes X \otimes A \nabla A$.

The partitioned matrices in (2.2) and (2.3) must be suitably interpreted in case $r = m$ or $r = n$.

Proof. Rewriting (2.2) as

$$A \nabla P^{\otimes^{-1}} \otimes \begin{pmatrix} E_r & C \\ \varepsilon & \varepsilon \end{pmatrix}$$

it is easily verified that any X given by (2.3) satisfies

$$A \otimes X \otimes A \nabla P^{\otimes^{-1}} \otimes \begin{pmatrix} E_r & C \\ \varepsilon & \varepsilon \end{pmatrix} \otimes \begin{pmatrix} E_r & \varepsilon \\ \varepsilon & D \end{pmatrix} \otimes P \otimes P^{\otimes^{-1}} \otimes \begin{pmatrix} E_r & C \\ \varepsilon & \varepsilon \end{pmatrix},$$

$$A \otimes X \otimes A \nabla P^{\otimes^{-1}} \otimes \begin{pmatrix} E_r & C \\ \varepsilon & \varepsilon \end{pmatrix} \otimes \begin{pmatrix} E_r & C \\ \varepsilon & \varepsilon \end{pmatrix} \nabla P^{\otimes^{-1}} \otimes \begin{pmatrix} E_r & C \\ \varepsilon & \varepsilon \end{pmatrix} \nabla A. \quad \square$$

Existence of the matrix X satisfying $A \otimes X \otimes A \nabla A$ is given by the following theorem:

Theorem 2.3. *Let* $A \in M_{m\times n}(\mathbb{S})$. *A matrix X satisfies* $A \otimes X \otimes A \nabla A$ *if and only if*

$$X \nabla \begin{pmatrix} E_r & \varepsilon \\ \varepsilon & D \end{pmatrix} \otimes P \tag{2.4}$$

for some D and for some nonbalanced P satisfying

$$P \otimes A \nabla \begin{pmatrix} E_r & C \\ \varepsilon & \varepsilon \end{pmatrix}. \tag{2.5}$$

Proof. ($\Leftarrow$) If (2.5) and (2.4) hold, then X satisfies $A \otimes X \otimes A \nabla A$ by Theorem 2.2.

($\Rightarrow$) Let $A \otimes X \otimes A \nabla A$. Then both $A \otimes X$ and $X \otimes A$ satisfy

$$A \otimes X \otimes A \otimes X \nabla A \otimes X \quad \text{and} \quad X \otimes A \otimes X \otimes A \nabla X \otimes A.$$

$A \otimes X$ and $X \otimes A$ have the same rank as A. Thus, both $A \otimes X$ and $X \otimes A$ are of the form $\begin{pmatrix} E_r & C \\ \varepsilon & \varepsilon \end{pmatrix}$. Therefore, there exists nonsingular R such that

$$R^{-1} \otimes A \otimes X \otimes R \nabla \begin{pmatrix} E_r & C \\ \varepsilon & \varepsilon \end{pmatrix}.$$

Thus,

$$R^{-1} \otimes A \nabla R^{-1} \otimes A \otimes X \otimes A,$$

$$R^{-1} \otimes A \nabla (R^{-1} \otimes A \otimes X \otimes R) \otimes R^{-1} \otimes A,$$

$$R^{-1} \otimes A \nabla \begin{pmatrix} E_r & C \\ \varepsilon & \varepsilon \end{pmatrix} \otimes R^{-1} \otimes A.$$

It follows that $R^{-1} \otimes A$ is of the form

$$R^{-1} \otimes A \nabla \begin{pmatrix} C^{-1} & E_r \\ \varepsilon & \varepsilon \end{pmatrix} \Leftrightarrow A \nabla R \otimes \begin{pmatrix} C^{-1} & E_r \\ \varepsilon & \varepsilon \end{pmatrix}.$$

Let $P = \begin{pmatrix} C & \varepsilon \\ \varepsilon & \varepsilon \end{pmatrix} \otimes R^{-1}$. Then

$$P \otimes A \nabla \begin{pmatrix} C & \varepsilon \\ \varepsilon & \varepsilon \end{pmatrix} \otimes R^{-1} \otimes R \otimes \begin{pmatrix} C^{-1} & E_r \\ \varepsilon & \varepsilon \end{pmatrix} \nabla \begin{pmatrix} E_r & C \\ \varepsilon & \varepsilon \end{pmatrix}.$$

Consider the matrix $X \otimes P^{-1}$. We have

$$X \otimes P^{-1} \otimes \begin{pmatrix} E_r & C \\ \varepsilon & \varepsilon \end{pmatrix} \nabla X \otimes P^{-1} \otimes P \otimes A \nabla X \otimes A.$$

So, $X \otimes P^{-1} \otimes \begin{pmatrix} E_r & C \\ \varepsilon & \varepsilon \end{pmatrix} \nabla \begin{pmatrix} E_r & C \\ \varepsilon & \varepsilon \end{pmatrix}$. From the latest equation, it follows that

$X \otimes P^{-1} \otimes \begin{pmatrix} E_r & C \\ \varepsilon & \varepsilon \end{pmatrix}$ for some D. We have $X \nabla \begin{pmatrix} E_r & C \\ \varepsilon & \varepsilon \end{pmatrix} \otimes P$. □

According to Theorem 2.3, we give the following example:

Example 2.4. *Let* $A = \begin{pmatrix} \ominus 2 & 1^{\bullet} & \varepsilon & 0 \\ \varepsilon & 1 & \varepsilon & \ominus 0 \\ 1 & 0^{\bullet} & 1 & \varepsilon \end{pmatrix}.$

We have $P \otimes A = A_3 \nabla E_A = \begin{pmatrix} e & \varepsilon & \varepsilon & \star \\ \varepsilon & e & \varepsilon & \star \\ \varepsilon & \varepsilon & e & \star \end{pmatrix} = (E_3 \quad C)$ with

$$P = E_{3(-1)} \otimes E_{32(0)} \otimes E_{2(-1)} \otimes E_{31(\ominus 1)} \otimes E_{1(\ominus(-2))},$$

$$P = \begin{pmatrix} e & \varepsilon & \varepsilon \\ \varepsilon & e & \varepsilon \\ \varepsilon & \varepsilon & -1 \end{pmatrix} \otimes \begin{pmatrix} e & \varepsilon & \varepsilon \\ \varepsilon & e & \varepsilon \\ \varepsilon & e & e \end{pmatrix} \otimes \begin{pmatrix} e & \varepsilon & \varepsilon \\ \varepsilon & -1 & \varepsilon \\ \varepsilon & \varepsilon & e \end{pmatrix}$$

$$\otimes \begin{pmatrix} e & \varepsilon & \varepsilon \\ \varepsilon & e & \varepsilon \\ \ominus 1 & \varepsilon & e \end{pmatrix} \otimes \begin{pmatrix} \ominus(-2) & \varepsilon & \varepsilon \\ \varepsilon & e & \varepsilon \\ \varepsilon & \varepsilon & e \end{pmatrix}.$$

So, we have $P = \begin{pmatrix} \ominus(-2) & \varepsilon & \varepsilon \\ \varepsilon & -1 & \varepsilon \\ -2 & -2 & -1 \end{pmatrix}.$

And, $P \otimes A = \begin{pmatrix} e & (-1)^{\bullet} & \varepsilon & \ominus(-2) \\ \varepsilon & e & \varepsilon & \ominus(-1) \\ 0^{\bullet} & (-1)^{\bullet} & e & (-2)^{\bullet} \end{pmatrix} \nabla (E_3 \quad C).$

There is

$$X = \begin{pmatrix} E_3 \\ \varepsilon \end{pmatrix} \otimes P = \begin{pmatrix} e & \varepsilon & \varepsilon \\ \varepsilon & e & \varepsilon \\ \varepsilon & \varepsilon & e \\ \varepsilon & \varepsilon & \varepsilon \end{pmatrix} \otimes \begin{pmatrix} \ominus(-2) & \varepsilon & \varepsilon \\ \varepsilon & -1 & \varepsilon \\ -2 & -2 & -1 \end{pmatrix} = \begin{pmatrix} \ominus(-2) & \varepsilon & \varepsilon \\ \varepsilon & -1 & \varepsilon \\ -2 & -2 & -1 \\ \varepsilon & \varepsilon & \varepsilon \end{pmatrix}$$

which satisfies $A \otimes X \otimes A \nabla A.$

Theorem 2.5. *Let* $A \in M_{m \times n}(\mathbb{S})$. *If* X *is any matrix satisfying* $A \otimes X \otimes A \nabla A$, *then* $A \otimes x \nabla b$ *has a solution if and only if* $A \otimes X \otimes b \nabla b$, *in which case the most general solution is* $x = X \otimes b \oplus (E \ominus X \otimes A) \otimes h$, *where h is arbitrary*.

Proof.

$$A \otimes x = A \otimes [X \otimes b \oplus (E \ominus X \otimes A) \otimes h],$$

$$A \otimes x = A \otimes X \otimes b \oplus A \otimes (E \ominus X \otimes A) \otimes h,$$

$$A \otimes x = (A \otimes X \otimes b) \oplus A \otimes h \ominus (A \otimes X \otimes A)$$

$$\otimes h \nabla b \oplus A \otimes h \ominus A \otimes h \nabla b \oplus (A \otimes h)^{\bullet}.$$

Because we have $(A \otimes h)^{\bullet} \nabla \varepsilon$, we conclude that $A \otimes x \nabla b \oplus \varepsilon = b$. Hence, $A \otimes x \nabla b.$ □

Corollary 2.6. *If* X *is any matrix satisfying* $A \otimes X \otimes A \nabla A$, *then* $A \otimes x \nabla \varepsilon$ *has a solution if and only if the most general solution is* $x = (E \ominus X \otimes A) \otimes h$, *where h is arbitrary*.

Proof.

$$A \otimes x = A \otimes (E \ominus X \otimes A) \otimes h = A \otimes h \ominus A \otimes X \otimes A \otimes h$$

$$= A \otimes h \ominus A \otimes h.$$

Because $A \otimes h \ominus A \otimes h = (A \otimes h)^{\bullet}$ and $(A \otimes h)^{\bullet} \nabla \varepsilon$, we conclude that $A \otimes x \nabla \varepsilon$. □

Corollary 2.7. *Vector* $x \nabla \begin{pmatrix} \ominus C \otimes y \\ y \end{pmatrix}$, *where y is arbitrary*, *is the general solution from linear balanced system* $A \otimes x \nabla \varepsilon$ *if and only if X that has* $\begin{pmatrix} E_r & \varepsilon \\ \varepsilon & D \end{pmatrix} \otimes P$ *form*, *where D is arbitrary*, *is any matrix satisfying* $A \otimes X \otimes A \nabla A$, *which* $A \nabla P^{\otimes -1} \otimes \begin{pmatrix} E_r & C \\ \varepsilon & \varepsilon \end{pmatrix}$.

Proof. According to Corollary 2.6, we have

$$x = (E \ominus X \otimes A) \otimes h \nabla \left[E \ominus \left(\left(\begin{pmatrix} E_r & \varepsilon \\ \varepsilon & D \end{pmatrix} \otimes P \right) \otimes \left(P^{\otimes -1} \otimes \begin{pmatrix} E_r & C \\ \varepsilon & \varepsilon \end{pmatrix} \right) \right) \right] \otimes h.$$

Furthermore, we obtain

$$x \nabla \left[E \ominus \begin{pmatrix} E_r & \varepsilon \\ \varepsilon & D \end{pmatrix} \otimes \begin{pmatrix} E_r & C \\ \varepsilon & \varepsilon \end{pmatrix} \right] \otimes h,$$

$$x \nabla \left[E \ominus \begin{pmatrix} E_r & C \\ \varepsilon & \varepsilon \end{pmatrix} \right] \otimes h.$$

If we take $E = \begin{pmatrix} E_r & \varepsilon \\ \varepsilon & E_{m-r} \end{pmatrix}$ and $h = \begin{pmatrix} h_r \\ h_{m-r} \end{pmatrix}$, then we obtain that x can be presented as the following form:

$$x \nabla \left[\begin{pmatrix} E_r & \varepsilon \\ \varepsilon & E_{m-r} \end{pmatrix} \otimes \begin{pmatrix} E_r & C \\ \varepsilon & \varepsilon \end{pmatrix} \right] \otimes \begin{pmatrix} h_r \\ h_{m-r} \end{pmatrix},$$

$$x \nabla \begin{pmatrix} \varepsilon & \ominus C \\ \varepsilon & E_{m-r} \end{pmatrix} \otimes \begin{pmatrix} h_r \\ h_{m-r} \end{pmatrix} \nabla \begin{pmatrix} \ominus C \otimes h_{m-r} \\ h_{m-r} \end{pmatrix}.$$

We now conclude that $x \nabla \begin{pmatrix} \ominus C \otimes y \\ y \end{pmatrix},$ where y is arbitrary, is the solution of $A \otimes x \nabla.$ □

Example 2.8. According to Example 2.4, because

$$P \otimes A = \begin{pmatrix} e & (-1)^{\bullet} & \varepsilon & \ominus(-2) \\ \varepsilon & e & \varepsilon & \ominus(-1) \\ 0^{\bullet} & (-1)^{\bullet} & e & (-2)^{\bullet} \end{pmatrix} \nabla (E_3 \quad C),$$

we have $C = \begin{pmatrix} \ominus(-2) \\ \ominus(-1) \\ (-2)^{\bullet} \end{pmatrix}.$ Hence, we get that $\ominus C = \begin{pmatrix} -2 \\ -1 \\ (-2)^{\bullet} \end{pmatrix}.$ If we take $y = 0,$ then we obtain $x = \begin{pmatrix} \ominus C \otimes y \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ (-2)^{\bullet} \\ 0 \end{pmatrix}$ is the solution of $A \otimes x \nabla \varepsilon.$

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